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ON THE FIELD OF A 2-BLOCK. II

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ABSTRACT. Every real 2-block B of a finite metabelian group contains an irreducible character θ such that $Q(B) = Q(\theta)$.

Let B be a p-block of a finite metabelian group G and let ζ be a primitive $|G|$th root of unity over the rationals Q. If $\tau \in \mathcal{G}(Q(\zeta)/Q)$, the Galois group, it was shown in [3] that the set of all χ^τ, where χ is an irreducible ordinary character of B, form all the irreducible ordinary characters of some block denoted by B^τ. Let $\mathcal{H}(B) = \{\tau \in \mathcal{G}(Q(\zeta)/Q) \mid B^\tau = B\}$ and $Q(B)$ the subfield of $Q(\zeta)$ fixed by $\mathcal{H}(B)$. Assuming B is a 2-block, it was proved in [3] that $Q(B) = Q(\theta)$ for some irreducible ordinary character $\theta \in B$ if the 2-Sylow subgroup of G' is cyclic (G' abelian). Such an equality does not hold in general. However, in this paper we prove that if B is real then $Q(B) = Q(\theta)$ for some irreducible $\theta \in B$. In particular, this gives another aspect of real 2-blocks studied in Gow [4], where it was proved that every real 2-block of any finite group has an irreducible real character.

THEOREM. *Let B be a real 2-block of a finite metabelian group G. Then B contains an irreducible 2-rational character θ of height zero such that $Q(B) = Q(\theta)$.*

PROOF. We use the results in [**1** and **2**]. Let P be the 2-Sylow subgroup of the commutator group G', $G' = P \times G_1$, and $H = P \times \Lambda$, $\Lambda \subseteq G_1$, such that G'/H is cyclic. For any subgroup L of G' let $K(L) \supseteq G'$ be a subgroup of $N(L)$, the normalizer of L in G, such that $K(L)/L$ is a maximal abelian subgroup of $N(L)/L$. Fix $K(\Lambda)$ and if $\Lambda \subseteq L \subseteq H$, pick $K(H) \supseteq K(L) \supseteq K(\Lambda)$. Let σ be a linear 2-modular representation of $K(\Lambda)$, $S = \ker\sigma$, and $S \cap G' = H$. Let $B(\sigma, H)$ be the collection of all representations T'^G where T' is a linear (ordinary) representation of $K(L)$, $\ker T' \cap G' = L$, with the modular representation $\bar{T}'_{K(\Lambda)}$ being G-conjugate to σ (notations are the same as in [2]), and L runs over all subgroups of G', G'/L cyclic, $\Lambda \subseteq L \subseteq H$. Include in $B(\sigma, H)$ the composition factors (and their Brauer characters) of the modular representations $\bar{T}'^G$, and the characters of T'^G. From [**2**, §4] $B(\sigma, H)$ is a 2-block and every 2-block of G is given in this form.

Fix H and σ and assume $B(\sigma, H)$ is real. If χ is the character afforded by $T'^G \in B(\sigma, H)$, then its complex conjugate $\chi^c \in B(\sigma, H)$. Thus $\bar{T}'_{K(\Lambda)}$ and $\overline{(T'^c)}_{K(\Lambda)}$ are G-conjugate to σ. Since $T'^c(k) = T'(k^{-1})$ for all $k \in K(L) \supseteq K(\Lambda)$, it follows

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that σ', defined by $\sigma'(f) = \sigma(f^{-1})$ for all $f \in K(\Lambda)$, is G-conjugate to σ. Since $\sigma'(f) = \sigma(f^{-1})$ means $\ker \sigma' = \ker \sigma$, there exists $y \in N(S)$, y unique modulo the inertia group $I(\sigma)$, such that $\sigma(y^{-1}fy) = \sigma(f^{-1})$ for all $f \in K(\Lambda)$, or y inverts every element of $K(\Lambda)/S$. Note that $K(\Lambda)/S$ is of odd order.

We claim that $K(H)/S$ splits over $K(\Lambda)/S$. Let r be a prime and R/S be the r-Sylow subgroup of $K(H)/S$, $K(H) \supseteq K(\Lambda)$, and $R_0/S = R/S \cap K(\Lambda)/S$. If $r = 2$ then $R_0/S = 1$. Thus let r be odd, $R_0/S = \langle cS \rangle$ where cS is of order r^s, and $R/R_0 = \langle z_1R_0, \ldots, z_nR_0 \rangle$, z_iR_0 being a basis for the abelian group R/R_0. Let $z = z_i$, and assume zR_0 is of order r^u with $z^{r^u} \equiv c^{r^v} \pmod S$. If $u \leqslant v$, then, letting $w = z^{-1}c^{r^{v-u}}$, we get $w^{r^u} \equiv 1 \pmod S$. Assume $v < s$ and $u > v$. Since $[R, y] \subseteq G' \cap R \subseteq K(\Lambda) \cap R = R_0$, it follows that $y^{-1}zy \equiv zc^{\lambda} \pmod S$. Thus

$$y^{-1}z^{r^u}y \equiv z^{r^u}c^{\lambda r^u} \equiv c^{-r^v} \pmod S,$$

implying $\lambda r^u \equiv -2r^v \pmod{r^s}$, a contradiction. Thus $K(H)/S$ splits over $K(\Lambda)/S$.

The above also implies that $cS, z_1S, \ldots, z_nS$, with the z_i's chosen such that z_iS is of order r^{u_i}, form a basis for R/S with $R_0/S = \langle cS \rangle$. (Note that if $r = 2$ then $R_0/S = 1$, and thus the above is true.) Let $y^{-1}z_iy \equiv z_ic^{\lambda_i} \pmod S$ and μ_i be a solution of $\lambda_i + 2\mu_i \equiv 0 \pmod{r^s}$. If $R_0/S = 1$ (as in the case of $r = 2$) let $\lambda_i = \mu_i = 0$. For each r-Sylow subgroup R/S of $K(H)/S$, we fix such a set of z_i's.

Our aim now is to construct the character θ of the theorem. Let T be a (complex) linear representation of $K(\Lambda)$, $\overline{T} = \sigma$, $\ker T \cap G' = H$, and $T(k) = 1$ for every 2-element k of $K(\Lambda)$. Then $\ker T = \ker \sigma = S$. Extend T as T' to all elements of $RK(\Lambda)$ for every r-Sylow subgroup $RK(\Lambda)/K(\Lambda)$ of $K(H)/K(\Lambda)$ and then to all elements of $K(H)$ as follows: If $R_0/S = 1$ (as in the case of $r = 2$) then let $T'(z_i) = 1$ for all i. If $R_0/S = \langle cS \rangle \neq 1$ (notations as above) let $T'(z_i) = T(c)^{\mu_i}$. Now if $k \in K(H)$ then $k = a(\Pi'(\Pi_i z_i^{e_i}))$, Π' running over all primes r dividing $|K(H)/S|$ and $a \in K(\Lambda)$. Define $T'(k) = T(a)(\Pi'(\Pi_i T'(z_i)^{e_i}))$. It is easy to show that T' is a linear representation of $K(H)$. Now $T' \in R(K(\Lambda), S, K(H))$, as in the notations of [**1**, p. 100], and hence T'^G is irreducible. Since $(\overline{T}')_{K(\Lambda)} = \sigma$ we have $T'^G \in B(\sigma, H)$. Let θ be the character of T'^G. Then θ is 2-rational and since $\ker \theta$ contains the defect group of $B(\sigma, H)$, θ is of height zero in $B(\sigma, H)$. It remains to show that $Q(B(\alpha, H)) = Q(\theta)$.

Now $I(T') = K(H)$, $I(T')$ the inertia group of T'. We claim that $I(\sigma) = K(H)$. Let $x \in I(\sigma)$. Then for any prime r, $T(x^{-1}cx) = T(c)$. (Here c, z_i depend on r as in the notations above.) If $R_0/S = 1$, then $x^{-1}zx = z \pmod S$ for any $z = z_i \in R$. Assume for some (odd) r, $R_0/S = \langle cS \rangle \neq 1$, and for some generator zS of R/S, $x^{-1}zx \equiv zc^{\delta} \pmod S$. Since $x^{-1}cx \equiv c \pmod S$, $x \neq y$. Thus we may assume $xy \equiv yxa \pmod S$, $a \in K(\Lambda)$. It follows that $(xy)^{-1}z(xy) \equiv (yx)^{-1}z(yx) \pmod S$. After a short computation we have $zc^{\lambda}c^{-\delta} \equiv zc^{\delta}c^{\lambda} \pmod S$, λ defined as above. Thus $2\delta \equiv 0 \pmod{r^s}$, and since $r \neq 2$, $\delta \equiv 0 \pmod{r^s}$ or $x \in K(H)$. Thus $I(\sigma) = K(H) = I(T')$.

Let ζ be a primitive $|G|$th root of unity. Then $Q(\chi) \subseteq Q(\zeta)$ for all irreducible characters χ in $B(\sigma, H)$. Let $\tau \in \mathcal{G}(Q(\zeta)/Q)$ and assume $B(\sigma, H)^{\tau} = B(\sigma, H)$. This implies both T'^G and $(T'^G)^\tau$ are in $B(\sigma, H)$ and thus $\sigma = \bar{T} = \bar{T}'_{K(\Lambda)}$ and $\bar{T}^\tau = (\bar{T}'^\tau)_{K(\Lambda)}$ are G-conjugate or T and T^τ are G-conjugate. Assume $T^\tau = T^g$ for some $g \in G$. Since $\ker T = \ker T^g = S$, it follows that $g \in N(S)$. Now, for the prime r, $r \neq 2$, the group $\langle y, g \rangle$ induces a (cyclic) automorphism subgroup on $R_0/S = \langle cS \rangle$. That is, there is an element $h_r \in N(S)$ such that $\langle h_r \rangle$ induces the same automorphism group on R_0/S as $\langle y, g \rangle$. If h_r induces an automorphism of order n on R_0/S then n is even and $h_r^{n/2}$ inverts every element of R_0/S. Assume $h_r^{-1} c h_r \equiv c^\alpha \pmod S$. Then $r \nmid \alpha - 1$, $g^{-1}cg \equiv c^{\alpha^t} \pmod S$ and $\alpha^{n/2} \equiv -1 \pmod{r^s}$. Assume $h_r^{-1} z h_r \equiv zc^\nu \pmod S$, $g^{-1}zg \equiv zc^\delta \pmod S$, where zS is a generator of R/S as defined above. Since

$$\left(yh_r^{n/2}\right)^{-1} z \left(yh_r^{n/2}\right) \equiv \left(h_r^{n/2}y\right)^{-1} z \left(h_r^{n/2}z\right) \pmod S,$$

$$\left(gh_r^t\right)^{-1} z \left(gh_r^t\right) \equiv \left(h_r^t g\right)^{-1} z \left(h_r^t g\right) \pmod S,$$

and

$$h_r^{-e} z h_r^e \equiv zc^{\nu(\alpha^{e-1} + \cdots + 1)} \pmod S \quad \text{for any } e,$$

it follows that

$$\lambda \equiv \nu\left(\alpha^{n/2-1} + \cdots + 1\right) \pmod{r^s} \quad \text{and} \quad \delta \equiv \nu\left(\alpha^{t-1} + \cdots + 1\right) \pmod{r^s}.$$

That is, $\langle y, g \rangle$ induces a cyclic automorphism group on R/S.

Recalling that $T'(z) = T(c)^\mu$, $\lambda + 2\mu \equiv 0 \pmod{r^s}$, and $T^\tau = T^g$, we have

$$T'(z)^\tau = \left[T(c)^\mu\right]^\tau = T(g^{-1}cg)^\mu = T(c)^{\mu\alpha^t}.$$

Also

$$T'(g^{-1}zg) = T'(zc^\delta) = T'(z)T(c)^\delta = T(c)^{\mu+\delta}.$$

We need to prove $\mu\alpha^t \equiv \mu + \delta \pmod{r^s}$ or $\mu(\alpha^t - 1) \equiv \delta \pmod{r^s}$. Since $\alpha^{n/2} \equiv -1 \pmod{r^s}$, we have

$$2\nu(\alpha^t - 1) \equiv -\nu(\alpha^t - 1)(\alpha^{n/2} - 1) \pmod{r^s}.$$

Since $r \nmid \alpha - 1$, we have

$$2\nu\left(\alpha^{t-1} + \cdots + 1\right) \equiv -\nu(\alpha^t - 1)\left(\alpha^{n/2} + \cdots + 1\right) \pmod{r^s}.$$

Thus

$$2\delta \equiv -\lambda(\alpha^t - 1) \pmod{r^s} \quad \text{or} \quad \delta \equiv \mu(\alpha^t - 1) \pmod{r^s}$$

as desired. Thus $T'^\tau = T'^g$ or T'^τ and T' are G-conjugate. This implies T'^G and $(T'^\tau)^G$ are equivalent or $\theta^\tau = \theta$. The proof is complete.

The above implies

COROLLARY. *Let B be a 2-block of a finite metabelian group and assume B contains all the Q-conjugates of some irreducible character. Then B contains an irreducible rational character of height zero.*

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