

## THE BURAU REPRESENTATION IS UNITARY

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**ABSTRACT.** A slight modification of the Burau representation of the braid group is shown to be unitary relative to an explicitly defined Hermitian form. This gives a partial answer to the problem of identifying the image of the Burau representation and provides a tool for attacking the question of whether or not the Burau representation is faithful.

In [2, problem 14, p. 217], Birman asked for a characterization of the image of the Burau representation

$$\beta: B_n \rightarrow GL_{n-1}(\mathbf{Z}[t, t^{-1}])$$

of the braid group  $B_n$ . We contribute the following partial answer to Birman's question, where, for  $M$  a matrix over  $\mathbf{Z}[t, t^{-1}]$ ,  $M^*$  denotes the conjugate-transpose of  $M$  and the conjugate of  $p(t)$  in  $\mathbf{Z}[t, t^{-1}]$  is defined to be  $p(t^{-1})$ .

**THEOREM.** *There exists a nonsingular  $(n-1) \times (n-1)$  matrix  $J_0$  over  $\mathbf{Z}[t, t^{-1}]$  such that for each  $w$  in  $B_n$  it follows that  $\beta(w)^* J_0 \beta(w) = J_0$ .*

(See §§1 and 2 below for notation.) View  $\mathbf{Z}[t, t^{-1}]$  as a subring of  $\mathbf{Z}[s, s^{-1}]$  where  $s^2 = t$ . Over  $\mathbf{Z}[s, s^{-1}]$ , a change-of-basis replaces  $J_0$  by a matrix  $J$  which is Hermitian:  $J = J^*$ . Thus, in the new basis, the Burau representation is unitary relative to the Hermitian form  $J$ . (Below, we actually deal with  $J$  first and derive the theorem as stated as a corollary.) Finally, we note that for each of the standard generators  $\sigma_i$  of  $B_n$ ,  $\beta(\sigma_i)$  is a unitary reflection. (See §3 for details.)

An important open question is whether or not the Burau representation is faithful. This is known for  $n \leq 3$  [4] (also see [2]). It seems likely that the results presented here will help answer this question for  $n \geq 4$ .

In §1 we establish some conventions about matrices and in §2 we describe the Burau representation. In §3 we prove the theorem and justify the remarks following its statement above. Finally, in §4 we offer two conjectures which, taken together, would imply that the Burau representation is faithful.

**1. Preliminaries.** Let  $\mathbf{Z}$  denote the ring of (rational) integers, let  $t$  be an indeterminate and let  $\mathbf{Z}[t, t^{-1}]$  denote the ring of Laurent polynomials with coefficients in  $\mathbf{Z}$ . We consider infinite-dimensional matrices  $A = (a_{ij})$  with entries in  $\mathbf{Z}[t, t^{-1}]$

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Received by the editors October 27, 1982 and, in revised form, June 24, 1983.

1980 *Mathematics Subject Classification.* Primary 20F36.

<sup>1</sup>This research was supported in part by the National Science Foundation under Grant No. MCS81-16327.

and indexed by  $(i, j)$  in  $\mathbf{Z} \times \mathbf{Z}$  such that each row and column of  $A$  contains only finitely many nonzero entries under usual matrix addition and multiplication. Let  $s$  be an indeterminate which satisfies  $s^2 = t$ . We view  $\mathbf{Z}[s, s^{-1}]$  as a sub-ring of  $\mathbf{Z}[t, t^{-1}]$  and also consider matrices over  $\mathbf{Z}[s, s^{-1}]$ .

Let  $I$  denote the identity matrix and for  $(i, j)$  in  $\mathbf{Z} \times \mathbf{Z}$  let  $e_{ij}$  denote the matrix whose  $(i, j)$  entry is 1 and all of whose other entries are 0. We use summation notation  $\sum$  indexed implicitly by  $\alpha$  in  $\mathbf{Z}$ . For example,  $I = \sum e_{\alpha\alpha}$ .

We also consider row and column vectors indexed by  $\mathbf{Z}$  with finitely many nonzero entries. Let  $e_i$  denote the column vector whose  $i$ th entry is 1 and all of whose other entries are 0.

For a Laurent polynomial  $p = p(t)$  define  $\bar{p}$  by  $\bar{p}(t) = p(t^{-1})$  and if  $A = (a_{ij})$  is a matrix define  $\bar{A}$  by  $\bar{A} = (\bar{a}_{ij})$ . Let  $A'$  denote the transpose of  $A$  and finally define  $A^* = (\bar{A})'$ . We will also use similar definitions for matrices defined over  $\mathbf{Z}[s, s^{-1}]$ .

**2. The Burau representation of  $B_n$ .** We view the braid group abstractly as having generators  $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$  and relations all appropriate

$$(1) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2.$$

According to [1],  $B_n$  may be viewed as the subgroup of the automorphism group of the free group  $F_n$  on  $x_1, x_2, \dots, x_n$  generated by the automorphisms

$$(2) \quad \sigma_i(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1}, & j = i, \\ x_i, & j = i + 1, \\ x_j, & \text{otherwise.} \end{cases}$$

Let  $F$  denote the free group on  $\{x_j | j \in \mathbf{Z}\}$  and for  $i \in \mathbf{Z}$ , let  $\sigma_i$  be the automorphism of  $F$  defined by (2). Let  $B$  denote the group of automorphisms of  $F$  generated by  $\{\sigma_i | i \in \mathbf{Z}\}$ . An easy consequence of [1] is that  $B$  has presentation with the given generators and all relations (1) above.

Let  $Z$  denote the infinite cyclic group with generator  $t$ , let  $z: F \rightarrow Z$  denote the homomorphism defined by  $z(x_i) = t$  and let  $K$  denote the kernel of  $z$ . Since each  $\sigma_i$  satisfies  $z\sigma_i = z$ ,  $B$  preserves  $K$ . Thus  $B$  acts on  $K/K'$  where  $K'$  denotes the commutator subgroup of  $K$ . Now (see [3]),  $K/K'$  may be identified with the free  $\mathbf{Z}[t, t^{-1}]$ -module on  $\{e_j | j \in \mathbf{Z}\}$  where  $e_j$  denotes the image of  $x_{j+1}x_j^{-1}$  in  $K/K'$ . In this notation,  $B$  acts on  $K/K'$  as follows:

$$(3) \quad \sigma_i(e_j) = \begin{cases} te_i + e_{i-1}, & j = i - 1, \\ -te_i, & j = i, \\ e_i + e_{i+1}, & j = i + 1, \\ e_j, & \text{otherwise.} \end{cases}$$

Thus, in the notation of §1,  $\sigma_i$  is given by the matrix

$$(4) \quad \beta(\sigma_i) = I + te_{ii-1} - (1 + t)e_{ii} + e_{ii+1}.$$

Equation (4) is the usual form of the Burau representation. We also consider a slight modification of (4): let  $P = \sum s^\alpha e_{\alpha\alpha}$  with  $s^2 = t$  as in §1 and let  $\beta_i = P^{-1}\beta(\sigma_i)P$ . Then easily

$$(5) \quad \beta_i = I + se_{ii-1} - (1 + s^2)e_{ii} + se_{ii+1}.$$

Clearly  $\beta_i^{-1} = \bar{\beta}_i$  and the analogous property holds for  $\beta(\sigma_i)$ .

**3. Proof of the theorem.** We are ready to prove that each  $\beta_i$  is unitary, relative to a Hermitian form which we now define. Let  $J$  be the matrix

$$(6) \quad J = (s + s^{-1})I - \sum (e_{\alpha-1\alpha} + e_{\alpha\alpha+1}),$$

and note that  $J = J^*$ . We prove that each  $\beta_i^* J \beta_i = J$ . Since  $\beta_i^{-1} = \bar{\beta}_i$  we have  $(\beta_i^*)^{-1} = \beta_i'$  and so it suffices to prove that  $J\beta_i = \beta_i' J$  or, in turn, that  $J(\beta_i - I) = (\beta_i' - I)J$ . (Recall that  $A'$  stands for the transpose of  $A$ .) The reader can easily verify that both sides of this equality are given by

$$\begin{pmatrix} -s & s^2 + 1 & -s \\ s^2 + 1 & -s(s + s^{-1})^2 & s^2 + 1 \\ -s & s^2 + 1 & -s \end{pmatrix}$$

centered at  $(i, i)$  and surrounded by 0's.

To prove the theorem (as stated) for  $B$ , let  $J_0 = P^*JP$ . Then clearly  $\beta(\sigma_i)^* J_0 \beta(\sigma_i) = J_0$ . Also, restricting from  $B$  to  $B_n$  is easy: if  $\mu < \nu$  let  $E_{\mu,\nu}$  denote the span of  $e_\mu, e_{\mu+1}, \dots, e_\nu$  over  $\mathbf{Z}[s, s^{-1}]$ . Then each  $\beta_i$  with  $\mu \leq i < \nu$  preserves  $E_{\mu,\nu}$  and hence respects the restriction of  $J$  to  $E_{\mu,\nu}$ . Finally from (5) it is clear that each  $\beta_i - I$  has rank 1 and thus  $\beta_i$  is a unitary reflection. To emphasize this fact we note the formula

$$(7) \quad \beta_i(v) = v - (s^2 + 1) \frac{\langle e_i, v \rangle}{\langle e_i, e_i \rangle} e_i$$

where for  $v$  and  $w$  in the linear span of  $\{e_i | i \in \mathbf{Z}\}$  over  $\mathbf{Z}[s, s^{-1}]$ ,  $\langle v, w \rangle$  denotes the Laurent polynomial  $v^*Jw$ .

**4. Two conjectures.** A more conventional unitary representation of  $B_n$  may be obtained by substituting for  $t$  any complex number of norm 1. Then  $\bar{p}$  becomes the usual complex conjugate of  $p$ . Let  $\tau_k$  denote a primitive  $k$ th root of unity and let  $\beta^{(k)}$  denote the representation obtained by substituting  $-\tau_k$  for  $t$ . Also, let  $B^{(k)}$  denote the quotient group of  $B$  obtained by setting each  $\sigma_i^k = 1$  and let  $N^{(k)}$  denote the kernel of the natural homomorphism  $B \rightarrow B^{(k)}$ . Here are two conjectures:

(C1)  $N^{(k)}$  is the kernel of  $\beta^{(k)}$ .

(C2) The intersection of the  $N^{(k)}$ 's is trivial.

These conjectures would clearly imply that the Burau representation is faithful and the theorem presented here should help to answer (C1).

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