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ON THE NUMBER OF LOCALLY BOUNDED FIELD TOPOLOGIES

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ABSTRACT. Kiltinen has proven that there exist $2^{|F|}$ first countable, locally bounded field topologies (the maximum number possible) on a field F of infinite transcendence degree over its prime subfield. We consider those fields F of countable transcendence degree over its prime subfield E. In particular it is shown that if the characteristic of F is zero and the transcendence degree of F over E is nonzero or if F is a field of prime characteristic and the transcendence degree of F over E is greater than one, then there exist $2^{|F|}$ normable, locally bounded field topologies on F.

1. Introduction and basic definitions. Let R be a ring and let $\mathfrak{T}$ be a ring topology on R, that is, $\mathfrak{T}$ is a topology on R making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to R. A subset A of R is *bounded* for $\mathfrak{T}$ if given any neighborhood U of zero, there exists a neighborhood V of zero such that $(VA) \cup (AV) \subseteq U$. $\mathfrak{T}$ is a *locally bounded topology* on R if there exists a fundamental system of neighborhoods of zero for $\mathfrak{T}$ consisting of bounded sets.

We recall that a *norm* N on a ring R is a function from R into the nonnegative reals satisfying $N(x) = 0$ if and only if $x = 0$, $N(x - y) \leq N(x) + N(y)$ and $N(xy) \leq N(x)N(y)$ for all x and y in R. If N is a norm on R, then $\{B_\varepsilon: \varepsilon > 0\}$ is a fundamental system of neighborhoods of zero for a Hausdorff, locally bounded topology $\mathfrak{T}_N$ on R where for each $\varepsilon > 0$, $B_\varepsilon = \{r \in R: N(r) < \varepsilon\}$. Two norms on R are *equivalent* if they define the same topology on R.

If N is a nontrivial norm on a field F, that is, $\mathfrak{T}_N$ is nondiscrete, then a subset A of F is bounded for $\mathfrak{T}_N$ if and only if A is bounded in norm. Furthermore, if N is a nontrivial norm on F, then $\mathfrak{T}_N$ is a field topology on F, that is, $\mathfrak{T}_N$ is a ring topology on F and the mapping $x \to x^{-1}$ from F^* to F^* is continuous as well. (The proof of this assertion is the same as that for normed algebras found on p. 75 of [**1**].) We shall make use of the following theorem proved by Cohn in [**3**, Theorem 6.1]: If $\mathfrak{T}$ is a Hausdorff locally bounded ring topology on a field F and if there exists a nonzero element c in F such that $\lim_{n\to\infty} c^n = 0$, then $\mathfrak{T}$ is normable. Hence by the previous remarks, $\mathfrak{T}$ is a Hausdorff, first countable, locally bounded field topology on F.

If F is any field, then there exist at most $2^{|F|}$ locally bounded ring topologies on F [**6**, Theorems 5 and 6]. In [**5**, proof of Theorem 2.1], Kiltinen proved that if F is a field of infinite transcendence degree over its prime subfield, then there exist $2^{|F|}$ first countable, locally bounded field topologies on F, the maximum number possible.

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The problem of determining the number of first countable locally bounded field topologies on a field F of finite transcendence degree over its prime subfield was first raised by Kiltinen in [**5**, p. 35] and again by Wieslaw in [**9**, p. 175]. In this paper we consider those fields F of countable transcendence degree over its prime subfield E. In particular it is shown that if the characteristic of F is zero and the transcendence degree of F over E is nonzero or if F is a field of prime characteristic and the transcendence degree of F over E is greater than one, then there exist $2^{|F|}$ normable, locally bounded field topologies on F.

2. Locally bounded field topologies.

LEMMA 1. *Let X be a set of cardinality $\aleph_0$. Then there exists a collection $\mathcal{E}$ of subsets of X satisfying:*
1. $|\mathcal{E}| = 2^{\aleph_0}$,
2. *if $E \in \mathcal{E}$, then $|E| = \aleph_0$,*
3. *if E_1 and E_2 are distinct elements of $\mathcal{E}$, then $|E_1 \setminus E_2| = \aleph_0$, and $|E_2 \setminus E_1| = \aleph_0$.*

PROOF. We may assume that X is the set of rational numbers. For each irrational number y, let $\langle r_i \rangle_{i=1}^{\infty}$ be a sequence of rational numbers converging to y in the usual topology on the reals and let $E_y = \{r_i: i = 1, 2, \dots\}$. If $y \neq z$, then $E_y \cap E_z$ is finite. Hence the set $\mathcal{E}$, defined by $\mathcal{E} = \{E_y: y \text{ irrational}\}$, satisfies properties 1–3.

(The author is grateful to Richard Hodel for simplifying her proof of Lemma 1.)

Let F be an infinite field and let $\langle F_n \rangle_{n=0}^{\infty}$ be a sequence of subrings of F. $\langle F_n \rangle_{n=0}^{\infty}$ is a *decomposition* of F if $1 \in F_0$, F_n is properly contained in F_{n+1} for all $n \geqslant 0$ and $F = \bigcup_{n=0}^{\infty} F_n$. Let $\mathcal{D}$ be a collection of decompositions of F. Define $\sim$ on $\mathcal{D}$ by $\langle F_n \rangle_{n=0}^{\infty} \sim \langle G_n \rangle_{n=0}^{\infty}$ if for each countable subset A of F, $A \subseteq F_N$ for some $N \geqslant 0$ if and only if $A \subseteq G_M$ for some $M \geqslant 0$. Clearly, $\sim$ is an equivalence relation on $\mathcal{D}$.

LEMMA 2. *Let F be a field and let E be its prime subfield. If the characteristic of F is zero or if the transcendence degree of F over E is nonzero, then there exists a collection $\mathcal{D}$ of pairwise inequivalent decompositions of F such that $|\mathcal{D}| = 2^{\aleph_0}$.*

PROOF. Suppose the transcendence degree of F over E is nonzero. Then there exists a subfield E_0 of F and a transcendental element x over E_0 such that F is an algebraic extension of $E_0(x)$. Let $p_1, p_2, \dots$ be a sequence of pairwise nonassociate, irreducible elements of $E_0[x]$ and for each $i \geqslant 1$, let $\hat{v}_{p_i}$ be an extension of the p_i-adic valuation from $E_0(x)$ to F. Let $A = \{p_{A,0}, p_{A,1}, \dots\}$ be any countably infinite subset of $\{p_1, p_2, \dots\}$. For each $n \geqslant 0$, let $F_{A,n} = \{a \in F: \hat{v}_{p_{A,i}}(a) \geqslant 0 \text{ for all } i \geqslant n\}$. Clearly, $1 \in F_{A,0}$, $F_{A,n}$ is a subring of F for all $n \geqslant 0$ and F_n is properly contained in F_{n+1} for all $n \geqslant 0$ as $p_{A,n}^{-1} \in F_{n+1} \setminus F_n$. Moreover, $F = \bigcup_{n=0}^{\infty} F_{A,n}$. So $\langle F_{A,n} \rangle_{n=0}^{\infty}$ is a decomposition of F. Let $\mathcal{E}$ be a collection of subsets of $\{p_1, p_2, \dots\}$ satisfying properties 1–3 of Lemma 1. If A and B are distinct element of $\mathcal{E}$ and $A \setminus B = \{q_i: i = 1, 2, \dots\}$, then $\{q_i^{-1}: i = 1, 2, \dots\} \subseteq F_{B,0}$ but $\{q_i^{-1}: i = 1, 2, \dots\}$ is not contained in $F_{A,n}$ for any $n \geqslant 0$. Thus $\langle F_{A,n} \rangle_{n=0}^{\infty}$ and $\langle F_{B,n} \rangle_{n=0}^{\infty}$ are inequivalent decompositions of F.

If the transcendence degree of F over E is zero, then F is a field of characteristic zero. Let $p_1, p_2, \ldots$ be a sequence of distinct, positive primes in $\mathbf{Z}$ and proceed as above.

Let F be an infinite field, let $\langle F_n \rangle_{n=0}^{\infty}$ be a decomposition of F and let x be a transcendental element over F. We may identify $F(x)$ with a subfield of the field of formal power series $F((x))$ over F. Define ϕ: $F \to \mathbf{N} \cup \{0\}$ by $\phi(a)$ is the smallest nonnegative integer n such that $a \in F_n$. Define $|\cdot\cdot|: F \to \mathbf{N} \cup \{0\}$ by

$$|a| = \begin{cases} 2^{\phi(a)} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Let $D = \{\Sigma a_i x^i \in F((x))\colon \lim_{i\to\infty} |a_i| 2^{-i} = 0\}$ and for each $\Sigma a_i x^i$ in D, let $N(\Sigma a_i x^i) = \sup_i |a_i| 2^{-i}$. Then D is a subfield of $F((x))$, N is a norm on D and D is the completion of $F(x)$ for the N-topology [**2**, Lemmas 2 and 3].

LEMMA 3. *Let x be a transcendental element over an infinite field F. If $\langle F_{1,n} \rangle_{n=0}^{\infty}$ and $\langle F_{2,n} \rangle_{n=0}^{\infty}$ are inequivalent decompositions of F, then there exist distinct, nondiscrete, normable locally bounded field topologies T_1 and T_2 on $F(x)$ corresponding to $\langle F_{1,n} \rangle_{n=0}^{\infty}$ and $\langle F_{2,n} \rangle_{n=0}^{\infty}$ respectively.*

PROOF. By the above remarks, there exist norms N_1 and N_2 on $F(x)$ corresponding to the decompositions $\langle F_{1,n} \rangle_{n=0}^{\infty}$ and $\langle F_{2,n} \rangle_{n=0}^{\infty}$, respectively. Let A be a subset of F such that $A \subseteq F_{1,n}$ for some $n \geqslant 0$ but $A \nsubseteq F_{2,m}$ for any $m \geqslant 0$. Then A is bounded in norm for N_1 but not for N_2. Consequently, the topologies defined on $F(x)$ by N_1 and N_2 are distinct.

THEOREM 1. *Let F be a field of characteristic zero, let E be its prime subfield and let $\mathfrak{B}$ be a transcendence base for F over E.*

1. *If $\mathfrak{B} = \phi$ and $[F\colon E] < \infty$, then there exist $\aleph_0$ first countable, locally bounded field topologies on F. Moreover each nondiscrete, Hausdorff, locally bounded field topology on F is normable and hence is first countable.*

2. *If $|\mathfrak{B}|$ is countable and nonzero, then there exist $2^{\aleph_0}$ normable, locally bounded field topologies on F.*

PROOF. 1 follows from Theorems 1.8 and 3.3 of [**7**]. We may therefore assume that $|\mathfrak{B}|$ is nonzero and countable. Thus $|F| = \aleph_0$ and so there exist at most $2^{\aleph_0}$ locally bounded ring topologies on F. Moreover, there exists a subfield E_0 of F and a transcendental element x over E_0 such that F is an algebraic extension of $E_0(x)$. By Lemmas 2 and 3, there exist $2^{\aleph_0}$ distinct, normable topologies on $E_0(x)$. By [**8**, Theorem 1.6], each locally bounded ring topology on $E_0(x)$ extends to a locally bounded ring topology on F. But if $\mathfrak{T}$ is a locally bounded ring topology on F whose restriction to $E_0(x)$ is defined by a nontrivial norm, then there exists a nonzero element c in $E_0(x)$ such that $c^n \to 0$ for $\mathfrak{T}$. Thus by Cohn's Theorem [**3**, Theorem 6.1], $\mathfrak{T}$ is normable and hence $\mathfrak{T}$ is a locally bounded field topology on F.

THEOREM 2. *Let F be a field of prime characteristic, let E be its prime subfield and let $\mathfrak{B}$ be a transcendence base for F over E.*

1. *If $\mathfrak{B} = \phi$, then there exist two first countable, locally bounded field topologies on F.*

2. *If $|\mathfrak{B}| = 1$ and $[F: E(\mathfrak{B})] < \infty$, then there exist $\aleph_0$ first countable, locally bounded field topologies on F. Moreover each nondiscrete, Hausdorff, locally bounded field topology on F is normable and hence first countable.*

3. *If $|\mathfrak{B}|$ is countable and greater than one, then there exist $2^{\aleph_0}$ normable, locally bounded field topologies on F.*

PROOF. By [**4**, Theorem 6.1], if F is an algebraic extension of E, then the only locally bounded ring topologies on F are the discrete and indiscrete topologies. The proof of 2 is the same as the proof of 1 of Theorem 1. If $|\mathfrak{B}| \geqslant 2$, let $x_1 \in \mathfrak{B}$ and let $E_0 = E(\mathfrak{B} \setminus \{x_1\})$. Then the transcendence degree of E_0 over E is nonzero. The proof that there exist $2^{\aleph_0}$ normable, locally bounded field topologies on F is the same as the proof of 2 of Theorem 1.

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