ON THE NUMBER OF LOCALLY BOUNDED FIELD TOPOLOGIES

JO-ANN D. COHEN

Abstract. Kiltinen has proven that there exist $2^{|F|}$ first countable, locally bounded field topologies (the maximum number possible) on a field $F$ of infinite transcendence degree over its prime subfield. We consider those fields $F$ of countable transcendence degree over its prime subfield $E$. In particular it is shown that if the characteristic of $F$ is zero and the transcendence degree of $F$ over $E$ is nonzero or if $F$ is a field of prime characteristic and the transcendence degree of $F$ over $E$ is greater than one, then there exist $2^{|F|}$ normable, locally bounded field topologies on $F$.

1. Introduction and basic definitions. Let $R$ be a ring and let $\mathcal{T}$ be a ring topology on $R$, that is, $\mathcal{T}$ is a topology on $R$ making $(x, y) \rightarrow x - y$ and $(x, y) \rightarrow xy$ continuous from $R \times R$ to $R$. A subset $A$ of $R$ is bounded for $\mathcal{T}$ if given any neighborhood $U$ of zero, there exists a neighborhood $V$ of zero such that $(VA) \cup (AV) \subseteq U$. $\mathcal{T}$ is a locally bounded topology on $R$ if there exists a fundamental system of neighborhoods of zero for $\mathcal{T}$ consisting of bounded sets.

We recall that a norm $N$ on a ring $R$ is a function from $R$ into the nonnegative reals satisfying $N(x) = 0$ if and only if $x = 0$, $N(x - y) \leq N(x) + N(y)$ and $N(xy) \leq N(x)N(y)$ for all $x$ and $y$ in $R$. If $N$ is a norm on $R$, then $\{B_\varepsilon : \varepsilon > 0\}$ is a fundamental system of neighborhoods of zero for a Hausdorff, locally bounded topology $\mathcal{T}_N$ on $R$ where for each $\varepsilon > 0$, $B_\varepsilon = \{r \in R : N(r) < \varepsilon\}$. Two norms on $R$ are equivalent if they define the same topology on $R$.

If $N$ is a nontrivial norm on a field $F$, that is, $\mathcal{T}_N$ is nondiscrete, then a subset $A$ of $F$ is bounded for $\mathcal{T}_N$ if and only if $A$ is bounded in norm. Furthermore, if $N$ is a nontrivial norm on $F$, then $\mathcal{T}_N$ is a field topology on $F$, that is, $\mathcal{T}_N$ is a ring topology on $F$ and the mapping $x \rightarrow x^{-1}$ from $F^*$ to $F^*$ is continuous as well. (The proof of this assertion is the same as that for normed algebras found on p. 75 of [1].) We shall make use of the following theorem proved by Cohn in [3, Theorem 6.1]: If $\mathcal{T}$ is a Hausdorff locally bounded ring topology on a field $F$ and if there exists a nonzero element $c$ in $F$ such that $\lim_{n \rightarrow \infty} c^t = 0$, then $\mathcal{T}$ is normable. Hence by the previous remarks, $\mathcal{T}$ is a Hausdorff, first countable, locally bounded field topology on $F$.

If $F$ is any field, then there exist at most $2^{|F|}$ locally bounded ring topologies on $F$ [6, Theorems 5 and 6]. In [5, proof of Theorem 2.1], Kiltinen proved that if $F$ is a field of infinite transcendence degree over its prime subfield, then there exist $2^{|F|}$ first countable, locally bounded field topologies on $F$, the maximum number possible.
The problem of determining the number of first countable locally bounded field topologies on a field $F$ of finite transcendence degree over its prime subfield was first raised by Kiltinen in [5, p. 35] and again by Wieslaw in [9, p. 175]. In this paper we consider those fields $F$ of countable transcendence degree over its prime subfield $E$. In particular it is shown that if the characteristic of $F$ is zero and the transcendence degree of $F$ over $E$ is nonzero or if $F$ is a field of prime characteristic and the transcendence degree of $F$ over $E$ is greater than one, then there exist $2^{|F|}$ normable, locally bounded field topologies on $F$.

2. Locally bounded field topologies.

**Lemma 1.** Let $X$ be a set of cardinality $\aleph_0$. Then there exists a collection $\mathcal{A}$ of subsets of $X$ satisfying:

1. $|\mathcal{A}| = 2^{\aleph_0}$,
2. If $E \in \mathcal{A}$, then $|E| = \aleph_0$,
3. If $E_1$ and $E_2$ are distinct elements of $\mathcal{A}$, then $|E_1 \setminus E_2| = \aleph_0$ and $|E_2 \setminus E_1| = \aleph_0$.

**Proof.** We may assume that $X$ is the set of rational numbers. For each irrational number $y$, let $\langle r_i \rangle_{i=1}^{\infty}$ be a sequence of rational numbers converging to $y$ in the usual topology on the reals and let $E_y = \{r_i: i = 1, 2, \ldots\}$. If $y \neq z$, then $E_y \cap E_z$ is finite. Hence the set $\mathcal{A}$, defined by $\mathcal{A} = \{E_y: y \text{ irrational}\}$, satisfies properties 1–3.

(The author is grateful to Richard Hodel for simplifying her proof of Lemma 1.)

Let $F$ be an infinite field and let $\langle F_n \rangle_{n=0}^{\infty}$ be a sequence of subrings of $F$. $\langle F_n \rangle_{n=0}^{\infty}$ is a *decomposition* of $F$ if $1 \in F_0$, $F_n$ is properly contained in $F_{n+1}$ for all $n \geq 0$ and $F = \bigcup_{n=0}^{\infty} F_n$. Let $\mathcal{G}$ be a collection of decompositions of $F$. Define $\sim$ on $\mathcal{G}$ by $\langle F_n \rangle_{n=0}^{\infty} \sim \langle G_n \rangle_{n=0}^{\infty}$ if for each countable subset $A$ of $F$, $A \subseteq F_N$ for some $N \geq 0$ if and only if $A \subseteq G_M$ for some $M \geq 0$. Clearly, $\sim$ is an equivalence relation on $\mathcal{G}$.

**Lemma 2.** Let $F$ be a field and let $E$ be its prime subfield. If the characteristic of $F$ is zero or if the transcendence degree of $F$ over $E$ is nonzero, then there exists a collection $\mathcal{G}$ of pairwise inequivalent decompositions of $F$ such that $|\mathcal{G}| = 2^{\aleph_0}$.

**Proof.** Suppose the transcendence degree of $F$ over $E$ is nonzero. Then there exists a subfield $E_0$ of $F$ and a transcendental element $x$ over $E_0$ such that $F$ is an algebraic extension of $E_0(x)$. Let $p_1, p_2, \ldots$ be a sequence of pairwise nonassociate, irreducible elements of $E_0[x]$ and for each $i \geq 1$, let $\delta_{p_i}$ be an extension of the $p_i$–adic valuation from $E_0(x)$ to $F$. Let $A = \{p_{A,0}, p_{A,1}, \ldots\}$ be any countably infinite subset of $\{p_1, p_2, \ldots\}$. For each $n \geq 0$, let $F_{A,n} = \{a \in F: \delta_{p_{A,n}}(a) \geq 0 \text{ for all } i \geq n\}$. Clearly, $1 \in F_{A,0}$, $F_{A,n}$ is a subring of $F$ for all $n \geq 0$ and $F_{A,n}$ is properly contained in $F_{A,n+1}$ for all $n \geq 0$ as $p_{A,n} \in F_{A,n+1} \setminus F_n$. Moreover, $F = \bigcup_{n=0}^{\infty} F_{A,n}$. So $\langle F_{A,n} \rangle_{n=0}^{\infty}$ is a decomposition of $F$. Let $\mathcal{G}$ be a collection of subsets of $\{p_1, p_2, \ldots\}$ satisfying properties 1–3 of Lemma 1. If $A$ and $B$ are distinct elements of $\mathcal{G}$ and $A \setminus B = \{q_i: i = 1, 2, \ldots\}$, then $\{q_i^{-1}: i = 1, 2, \ldots\} \subseteq F_{B,0}$ but $\{q_i^{-1}: i = 1, 2, \ldots\}$ is not contained in $F_{A,n}$ for any $n \geq 0$. Thus $\langle F_{A,n} \rangle_{n=0}^{\infty}$ and $\langle F_{B,n} \rangle_{n=0}^{\infty}$ are inequivalent decompositions of $F$.
If the transcendence degree of $F$ over $E$ is zero, then $F$ is a field of characteristic zero. Let $p_1, p_2, \ldots$ be a sequence of distinct, positive primes in $\mathbb{Z}$ and proceed as above.

Let $F$ be an infinite field, let $\langle F_n \rangle_{n=0}^\infty$ be a decomposition of $F$ and let $x$ be a transcendental element over $F$. We may identify $F(x)$ with a subfield of the field of formal power series $F((x))$ over $F$. Define $\phi: F \to \mathbb{N} \cup \{0\}$ by $\phi(a)$ is the smallest nonnegative integer $n$ such that $a \in F_n$. Define $|\cdot|: F \to \mathbb{N} \cup \{0\}$ by

$$
|a| = \begin{cases} 2^{\phi(a)} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. 
\end{cases}
$$

Let $D = \{\sum a_i x^i \in F((x)) : \lim_{i \to \infty} |a_i| 2^{-i} = 0\}$ and for each $\sum a_i x^i$ in $D$, let $N(\sum a_i x^i) = \sup_{i} |a_i| 2^{-i}$. Then $D$ is a subfield of $F((x))$, $N$ is a norm on $D$ and $D$ is the completion of $F(x)$ for the $N$-topology [2, Lemmas 2 and 3].

**Lemma 3.** Let $x$ be a transcendental element over an infinite field $F$. If $\langle F_{1,n} \rangle_{n=0}^\infty$ and $\langle F_{2,n} \rangle_{n=0}^\infty$ are inequivalent decompositions of $F$, then there exist distinct, nondiscrete, normable locally bounded field topologies $T_1$ and $T_2$ on $F(x)$ corresponding to $\langle F_{1,n} \rangle_{n=0}^\infty$ and $\langle F_{2,n} \rangle_{n=0}^\infty$ respectively.

**Proof.** By the above remarks, there exist norms $N_1$ and $N_2$ on $F(x)$ corresponding to the decompositions $\langle F_{1,n} \rangle_{n=0}^\infty$ and $\langle F_{2,n} \rangle_{n=0}^\infty$, respectively. Let $A$ be a subset of $F$ such that $A \subseteq F_{1,n}$ for some $n > 0$ but $A \nsubseteq F_{2,m}$ for any $m > 0$. Then $A$ is bounded in norm for $N_1$ but not for $N_2$. Consequently, the topologies defined on $F(x)$ by $N_1$ and $N_2$ are distinct.

**Theorem 1.** Let $F$ be a field of characteristic zero, let $E$ be its prime subfield and let $\mathfrak{B}$ be a transcendence base for $F$ over $E$.

1. If $|\mathfrak{B}| = \phi$ and $[F: E] < \infty$, then there exist $2^{\aleph_0}$ first countable, locally bounded field topologies on $F$. Moreover each nondiscrete, Hausdorff, locally bounded field topology on $F$ is normable and hence is first countable.

2. If $|\mathfrak{B}|$ is countable and nonzero, then there exist $2^{\aleph_0}$ normable, locally bounded field topologies on $F$.

**Proof.** 1 follows from Theorems 1.8 and 3.3 of [7]. We may therefore assume that $|\mathfrak{B}|$ is nonzero and countable. Thus $|F| = 2^{\aleph_0}$ and so there exist at most $2^{\aleph_0}$ locally bounded ring topologies on $F$. Moreover, there exists a subfield $E_0$ of $F$ and a transcendental element $x$ over $E_0$ such that $F$ is an algebraic extension of $E_0(x)$. By Lemmas 2 and 3, there exist $2^{\aleph_0}$ distinct, normable topologies on $E_0(x)$. By [8, Theorem 1.6], each locally bounded ring topology on $E_0(x)$ extends to a locally bounded ring topology on $F$. But if $\mathfrak{T}$ is a locally bounded ring topology on $F$ whose restriction to $E_0(x)$ is defined by a nontrivial norm, then there exists a nonzero element $c$ in $E_0(x)$ such that $c^n \to 0$ for $\mathfrak{T}$. Thus by Cohn’s Theorem [3, Theorem 6.1], $\mathfrak{T}$ is normable and hence $\mathfrak{T}$ is a locally bounded field topology on $F$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
**Theorem 2.** Let $F$ be a field of prime characteristic, let $E$ be its prime subfield and let $\mathfrak{B}$ be a transcendence base for $F$ over $E$.

1. If $|\mathfrak{B}| = \phi$, then there exist two first countable, locally bounded field topologies on $F$.

2. If $|\mathfrak{B}| = 1$ and $[F: E(\mathfrak{B})] < \infty$, then there exist $\aleph_0$ first countable, locally bounded field topologies on $F$. Moreover each nondiscrete, Hausdorff, locally bounded field topology on $F$ is normable and hence first countable.

3. If $|\mathfrak{B}|$ is countable and greater than one, then there exist $2^{\aleph_0}$ normable, locally bounded field topologies on $F$.

**Proof.** By [4, Theorem 6.1], if $F$ is an algebraic extension of $E$, then the only locally bounded ring topologies on $F$ are the discrete and indiscrete topologies. The proof of 2 is the same as the proof of 1 of Theorem 1. If $|\mathfrak{B}| \geq 2$, let $x_1 \in \mathfrak{B}$ and let $E_0 = E(\mathfrak{B}\setminus\{x_1\})$. Then the transcendence degree of $E_0$ over $E$ is nonzero. The proof that there exist $2^{\aleph_0}$ normable, locally bounded field topologies on $F$ is the same as the proof of 2 of Theorem 1.

**References**


DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27650

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use