

AN ALGEBRAIC PERIODICITY THEOREM FOR SPHERES

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ABSTRACT. A periodicity theorem is given for spheres over a field of finite level generalizing results of Jouanolou. An extension of this result gives families of smooth affine varieties with isomorphic K -groups.

The Bott periodicity theorem [1] lies at the foundation of topological K -theory. In particular, it yields the fact that the real and complex topological K -groups of spheres are periodic in the dimension. Consequent to work of Fossum [2] and Jouanolou [4], it was noted that the algebraic K_0 -groups of complex spheres were isomorphic to the topological K -groups hence they enjoyed periodic behavior. Later Jouanolou [5] showed that if -1 is a square in the field k and if $X_n = \text{Spec}(k[x_0, \dots, x_n]/(1 - \sum_{j=0}^n x_j^2))$ then $K_i(X_n) \simeq K_i(k) \oplus K_i(k)$ if n is even while $K_i(k) \oplus K_{i-1}(k)$ if n is odd where i is arbitrary and $K_i(k) = K_i(\text{Spec } k)$.

In this paper we prove the following "algebraic periodicity" theorem which is an extension of Jouanolou's result. If k is a field of characteristic not equal to two, $l(k)$, the level of the field k , is finite and X_n as above, then $K_i(X_n) \simeq K_i(X_{n+2l(k)})$. This is Jouanolou's result when -1 is a square in k .

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Consider the Witt ring $W(k)$ of a field k [7]. The characteristic of this ring can be characterized by the concept of the level of a field. The level of a field k , $l(k)$, is the smallest integer n such that -1 is the sum of n squares.

PROPOSITION [7]. *The characteristic of $W(k)$ is $2l(k)$ provided $l(k) < \infty$ and if $l(k) = \infty$ then the characteristic of $W(k) = 0$.*

THEOREM [8]. *The level of a field is a power of two or infinite.*

For a full discussion of the definitions and notations of K -theory we refer the reader to the papers of Quillen [9, 10] and Grayson [3]. We will, however, use the G_* -notation for the K'_* -notation of [10], and remark that the following Lemma does not seem to appear in the literature.

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LEMMA. *Let X be a separated Noetherian k -scheme with a k -rational point; then $K_i(k)$ is a direct summand of $K_i(X)$. If, moreover, X is smooth projective, then $G_i(k) \simeq K_i(k)$ is a direct summand of $G_i(X)$.*

PROOF. Let $\tau: X \rightarrow \text{Spec } k$ be the structure map and $\lambda: \text{Spec } k \rightarrow X$ the imbedding of the rational point. Then the composition $\tau \circ \lambda$ is the identity on $\text{Spec } k$. Hence the induced maps under K_* give a splitting. If X is smooth then $K_i(X) \simeq G_i(X)$, while if X is projective then λ, τ and $\tau \circ \lambda$ are proper; hence, using [10, §7, 2.7] the induced maps under G_* give a splitting.

For X a scheme with k -rational point x and imbedding λ , denote the kernel of the map $\lambda^*: K_*(X) \rightarrow K_*(\text{Spec } k)$ by $T_x K_*(X)$ (resp. denote the kernel of the map $\lambda^*: G_*(X) \rightarrow G_*(\text{Spec } k)$ by $T_x G_*(X)$). Note that when X is a k -scheme the isomorphism class of $T_x K_i(X)$ is independent of x and we denote it by $TK_i(X)$ so that $K_i(X) \simeq K_i(k) \oplus TK_i(X)$. If further X is smooth or projective then the isomorphism class of $T_x G_i(X)$ is independent of x and we denote it by $TG_i(X)$ so that $G_i(X) \simeq G_i(k) \oplus TG_i(X)$.

We are now in a position to address ourselves to the principal Theorem.

THEOREM. *Suppose that k is a field of characteristic not equal to two, that $l(k) < \infty$, and let $X_n = \text{Spec}(k[x_0, \dots, x_n]/(1 - \sum_{i=0}^n x_i^2))$. Then $K_i(X_n) \simeq K_i(X_{n+2l(k)})$.*

PROOF. For all n , X_n is regular, hence $K_i(X_n) \simeq G_i(X_n)$. As $\text{char } W(k) = 2l(k)$ we have by the results on quadratic forms that

$$X_{n+2l(k)} \simeq \text{Spec} \left(k[x_0, \dots, x_n, u_1, \dots, u_{l(k)}, v_1, \dots, v_{l(k)}] / \left(1 - \sum_{i=0}^n x_i^2 - \sum_{j=0}^{l(k)} u_j v_j \right) \right).$$

For the sake of convenience let us denote

$$Z_\lambda = \text{Spec} \left(k[x_0, \dots, x_n, u_1, \dots, u_{l(k)}, v_1, \dots, v_\lambda] / \left(1 - \sum_{i=0}^n x_i^2 - \sum_{j=1}^\lambda u_j v_j \right) \right) \lambda \geq 1$$

and let

$$Z_0 = \text{Spec} \left(k[x_0, \dots, x_n, u_1, \dots, u_{l(k)}] / \left(1 - \sum_{i=0}^n x_i^2 \right) \right) \simeq X_n[u_1, \dots, u_{l(k)}],$$

and let

$$U_\lambda = \text{Spec} \left(k[x_0, \dots, x_n, u, \dots, u_\lambda, \dots, u_{l(k)}, v_1, \dots, v_\lambda, v_\lambda^{-1}] \right) \text{ for } \lambda \geq 1.$$

We note that Z_λ and U_λ are regular. Then by Quillen's localization theorem (see [3, p. 299; 9, Theorem 4; 10, §7, Proposition 3.2]), we have for $f: Z_\lambda \rightarrow Z_{\lambda+1}$ and $j: U_{\lambda+1} \rightarrow Z_{\lambda+1}$ the natural immersions, the long exact sequence

$$(*) \quad \cdots \xrightarrow{\partial} K_i(Z_\lambda) \xrightarrow{f^*} K_i(Z_{\lambda+1}) \xrightarrow{j^*} K_i(U_{\lambda+1}) \xrightarrow{\partial} K_{i-1}(Z_\lambda) \xrightarrow{f^*} K_{i-1}(Z_{\lambda+1}) \xrightarrow{j^*} \cdots$$

Z_λ is a k -scheme with a k -rational point. Let $\tau: Z_\lambda \rightarrow \text{Spec } k$ be the structure map and let $\pi: \text{Spec } k \rightarrow Z_\lambda$ be the imbedding. U_λ is also a k -scheme with a k -rational point; let $\gamma: \text{Spec } k \rightarrow U_\lambda$ be the imbedding. Further, by the fundamental theorem of K -theory [3, 10], $K_i(U_\lambda) \simeq K_i(k) \oplus K_{i-1}(k)$.

The composition $\text{Spec } k \xrightarrow{\gamma} U_{\lambda+1} \xrightarrow{j} Z_{\lambda+1} \xrightarrow{\tau} \text{Spec } k$ is the identity on $\text{Spec } k$, hence $j^*|_{K_i(k)}$ is an isomorphism onto the complement of $K_{i-1}(k) (= TK_i(U_{\lambda+1}))$. Moreover, the composition

$$\text{Spec } K \xrightarrow{\pi} Z_\lambda \xrightarrow{f} Z_{\lambda+1} \xrightarrow{\tau} \text{Spec } k$$

is the identity on $\text{Spec } k$, hence $f^*|_{K_i(k)}$ is an isomorphism onto the complement of $TK_i(Z_\lambda)$. Kato's theorem on boundary map [6, §2.4] says that $\partial|_{K_i(k)} = f^*|_{K_i(k)}$, hence ∂ maps onto the complement of $TK_i(Z_\lambda)$. Therefore we have that $TK_i(Z_\lambda) \simeq TK_i(Z_{\lambda+1})$ by restriction of the map f_* . So $K_i(Z_\lambda) \simeq K_i(Z_{\lambda+1})$ for all λ and i . But $K_i(Z_0) \simeq K_i(X_n)$ by the fundamental theorem [3, 10] while $Z_{i(k)} \simeq X_{n+2l(k)}$. So $K_i(X_{n+2l(k)}) \simeq K_i(X_n)$.

An examination of the above proof indicates that we have proved more

THEOREM. *Let $Y = \text{Spec}(k[x_0, \dots, x_n]/p(\mathbf{x}))$ be regular and let*

$$V = \text{Spec}(k[x_0, \dots, x_n, u, v]/p(\mathbf{x}) + uv).$$

Then $K_i(Y) \simeq K_i(V)$.

We immediately obtain

COROLLARY. (1) *If $l(k) \neq \infty$ and $n \equiv 0 \pmod{2l(k)}$ then $K_i(X_n) \simeq K_i(k) \oplus K_i(k)$*
 (2) *If $l(k) \neq \infty$ and $n \equiv -1 \pmod{2l(k)}$ then $K_i(X_n) \simeq K_i(k) \oplus K_{i-1}(k)$.*

PROOF. (1) By induction it is enough to show this is true for $n = 0$. But $X_0 = \text{Spec}(k[x_0]/(1 - x_0^2))$ and $k[x_0]/1 - x_0^2 \simeq k \oplus k$, hence $K_i(X_0) \simeq K_i(k) \oplus K_i(k)$.

(2) Again by induction it is enough to show this is true for $n = 2l(k) - 1$. But $X_{2l(k)-1} \simeq \text{Spec}(k[u_1, \dots, u_{l(k)}, v_1, \dots, v_{l(k)}]/(1 - \sum_{j=1}^{l(k)} u_j v_j))$, hence

$$\begin{aligned} K_1(X_{2l(k)-1}) &\simeq K_i(\text{Spec } k[u, v]/(1 - uv)) \\ &\simeq K_i(\text{Spec } k[u, u^{-1}]) \simeq K_i(k) \oplus K_{i-1}(k). \end{aligned}$$

This Corollary is the result proved by Jouanolou [5].

During the preparation of this paper, the author has become aware of recent work by A. Suslin [11, Theorem 2.3] proving part (2) of the Corollary by similar methods.

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