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AN ALGEBRAIC PERIODICITY THEOREM FOR SPHERES

JOSEPH P. BRENNAN[1]

ABSTRACT. A periodicity theorem is given for spheres over a field of finite level generalizing results of Jouanolou. An extension of this result gives families of smooth affine varities with isomorphic K-groups.

The Bott periodicity theorem [1] lies at the foundation of topological K-theory. In particular, it yields the fact that the real and complex topological K-groups of spheres are periodic in the dimension. Consequent to work of Fossum [2] and Jouanolou [4], it was noted that the algebraic K_0-groups of complex spheres were isomorphic to the topological K-groups hence they enjoyed periodic behavior. Later Jouanolou [5] showed that if -1 is a square in the field k and if $X_n = \operatorname{Spec}(k[x_0,\ldots,x_n]/(1 - \Sigma_{j=0}^n x_j^2))$ then $K_i(X_n) \simeq K_i(k) \oplus K_i(k)$ if n is even while $K_i(k) \oplus K_{i-1}(k)$ if n is odd where i is arbitrary and $K_i(k) = K_i(\operatorname{Spec} k)$.

In this paper we prove the following "algebraic periodicity" theorem which is an extension of Jouanolou's result. If k is a field of characteristic not equal to two, $l(k)$, the level of the field k, is finite and X_n as above, then $K_i(X_n) \simeq K_i(X_{n+2l(k)})$. This is Jouanolou's result when -1 is a square in k.

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Consider the Witt ring $W(k)$ of a field k [7]. The characteristic of this ring can be characterized by the concept of the level of a field. The level of a field k, $l(k)$, is the smallest integer n such that -1 is the sum of n squares.

PROPOSITION [7]. *The characteristic of* $W(k)$ *is* $2l(k)$ *provided* $l(k) < \infty$ *and if* $l(k) = \infty$ *then the characteristic of* $W(k) = 0$.

THEOREM [8]. *The level of a field is a power of two or infinite.*

For a full discussion of the definitions and notations of K-theory we refer the reader to the papers of Quillen [9, 10] and Grayson [3]. We will, however, use the G_*-notation for the K'_*-notation of [10], and remark that the following Lemma does not seem to appear in the literature.

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LEMMA. *Let X be a separated Noetherian k-scheme with a k-rational point; then $K_i(k)$ is a direct summand of $K_i(X)$. If, moreover, X is smooth projective, then $G_i(k) \simeq K_i(k)$ is a direct summand of $G_i(X)$.*

PROOF. Let τ: $X \to \operatorname{Spec} k$ be the structure map and τ: $\operatorname{Spec} k \to X$ the imbedding of the rational point. Then the composition $\tau \circ \lambda$ is the identity on $\operatorname{Spec} k$. Hence the induced maps under K_* give a splitting. If X is smooth then $K_i(X) \simeq G_i(X)$, while if X is projective then λ, τ and $\tau \circ \lambda$ are proper; hence, using [**10**, §7, 2.7] the induced maps under G_* give a splitting.

For X a scheme with k-rational point $\mathbf{x}$ and imbedding λ, denote the kernel of the map λ^*: $K_*(X) \to K_*(\operatorname{Spec} k)$ by $T_{\mathbf{x}}K_*(X)$ (resp. denote the kernel of the map λ^*: $G_*(X) \to G_*(\operatorname{Spec} k)$ by $T_{\mathbf{x}}G_*(X)$). Note that when X is a k-scheme the isomorphism class of $T_{\mathbf{x}}K_i(X)$ is independent of $\mathbf{x}$ and we denote it by $TK_i(X)$ so that $K_i(X) \simeq K_i(k) \oplus TK_i(X)$. If further X is smooth or projective then the isomorphism class of $T_{\mathbf{x}}G_i(X)$ is independent of x and we denote it by $TG_i(X)$ so that $G_i(X) \simeq G_i(k) \oplus TG_i(X)$.

We are now in a position to address ourselves to the principal Theorem.

THEOREM. *Suppose that k is a field of characteristic not equal to two, that $l(k) < \infty$, and let $X_n = \operatorname{Spec}(k[x_0,\ldots,x_n]/(1 - \Sigma_{i=0}^n x_i^2))$. Then $K_i(X_n) \simeq K_i(X_{n+2l(k)})$.*

PROOF. For all n, X_n is regular, hence $K_i(X_n) \simeq G_i(X_n)$. As char $W(k) = 2l(k)$ we have by the results on quadratic forms that

$$X_{n+2l(k)} \simeq \operatorname{Spec}\left(k\left[x_0,\ldots,x_n, u_1,\ldots,u_{l(k)}, v_1,\ldots,v_{l(k)}\right]/\left(1 - \sum_{i=0}^{n} x_1^2 - \sum_{j=0}^{l(k)} u_j v_j\right)\right).$$

For the sake of convenience let us denote

$$Z_\lambda = \operatorname{Spec}\left(k\left[x_0,\ldots,x_n, u_1,\ldots,u_{l(k)}, v_1,\ldots,v_\lambda\right]/\left(1 - \sum_{i=0}^{n} x_1^2 - \sum_{j=1}^{\lambda} u_j v_j\right)\right) \lambda \geqslant 1$$

and let

$$Z_0 = \operatorname{Spec}\left(k\left[x_0,\ldots,x_n, u_1,\ldots,u_{l(k)}\right]/\left(1 - \sum_{i=0}^{n} x_1^2\right)\right) \simeq X_n\left[u_1,\ldots,u_{l(k)}\right],$$

and let

$$U_\lambda = \operatorname{Spec}\left(k\left[x_0,\ldots,x_n, u,\ldots,u_\lambda,\ldots,u_{l(k)}, v_1,\ldots,v_\lambda, v_\lambda^{-1}\right]\right) \quad \text{for } \lambda \geqslant 1.$$

We note that Z_λ and U_λ are regular. Then by Quillen's localization theorem (see [**3**, p. 299; **9**, Theorem 4; **10**, §7, Proposition 3.2]), we have for f: $Z_\lambda \to Z_{\lambda+1}$ and j: $U_{\lambda+1} \to Z_{\lambda+1}$ the natural immersions, the long exact sequence

$$(*) \qquad \cdots \xrightarrow{\partial} K_i(Z_\lambda) \xrightarrow{f_*} K_i(Z_{\lambda+1}) \xrightarrow{j^*} K_i(U_{\lambda+1}) \xrightarrow{\partial} K_{i-1}(Z_\lambda) \xrightarrow{f_*} K_{i-1}(Z_{\lambda+1}) \xrightarrow{j^*}.$$

Z_λ is a k-scheme with a k-rational point. Let τ: $Z_\lambda \to \operatorname{Spec} k$ be the structure map and let π: $\operatorname{Spec} k \to Z_\lambda$ be the imbedding. U_λ is also a k-scheme with a k-rational point; let γ: $\operatorname{Spec} k \to U_\lambda$ be the imbedding. Further, by the fundamental theorem of K-theory [**3**, **10**], $K_i(U_\lambda) \simeq K_i(k) \oplus K_{i-1}(k)$.

The composition $\operatorname{Spec} k \xrightarrow{\gamma} U_{\lambda+1} \xrightarrow{j} Z_{\lambda+1} \xrightarrow{\tau} \operatorname{Spec} k$ is the identity on Spec k, hence $j^* |_{K_i(k)}$ is an isomorphism onto the complement of $K_{i-1}(k)$ $(= TK_i(U_{\lambda+1}))$. Moreover, the composition

$$\operatorname{Spec} K \xrightarrow{\pi} Z_\lambda \xrightarrow{f} Z_{\lambda+1} \xrightarrow{\tau} \operatorname{Spec} k$$

is the identity on Spec k, hence $f^* |_{K_i(k)}$ is an isomorphism onto the complement of $TK_i(Z_\lambda)$. Kato's theorem on boundary map [**6**, §2.4] says that $\partial |_{K_i(k)} = f^* |_{K_i(k)}$, hence ∂ maps onto the complement of $TK_i(Z_\lambda)$. Therefore we have that $TK_i(Z_\lambda) \simeq TK_i(Z_{\lambda+1})$ by restriction of the map f_*. So $K_i(Z_\lambda) \simeq K_i(Z_{\lambda+1})$ for all λ and i. But $K_i(Z_0) \simeq K_i(X_n)$ by the fundamental theorem [**3, 10**] while $Z_{l(k)} \simeq X_{n+2l(k)}$. So $K_i(X_{n+2l(k)}) \simeq K_i(X_n)$.

An examination of the above proof indicates that we have proved more

THEOREM. *Let* $Y = \operatorname{Spec}(k[x_0,\ldots,x_n]/p(\mathbf{x}))$ *be regular and let*

$$V = \operatorname{Spec}(k[x_0,\ldots,x_n u, v]/p(\mathbf{x}) + uv).$$

Then $K_i(Y) \simeq K_i(V)$.

We immediately obtain

COROLLARY. (1) *If* $l(k) \neq \infty$ *and* $n \equiv 0 \bmod 2l(k)$ *then* $K_i(X_n) \simeq K_i(k) \oplus K_i(k)$
(2) *If* $l(k) \neq \infty$ *and* $n \equiv -1 \bmod 2l(k)$ *then* $K_i(X_n) \simeq K_i(k) \oplus K_{i-1}(k)$.

PROOF. (1) By induction it is enough to show this is true for $n = 0$. But $X_0 = \operatorname{Spec}(k[x_0]/(1 - x_0^2))$ and $k[x_0]/1 - x_0^2 \simeq k \oplus k$, hence $K_i(X_0) \simeq K_i(k) \oplus K_i(k)$.

(2) Again by induction it is enough to show this is true for $n = 2l(k) - 1$. But $X_{2l(k)-1} \simeq \operatorname{Spec}(k[u_1,\ldots,u_{l(k)}, v_1,\ldots,v_{l(k)}]/(1 - \Sigma_{j=1}^{l(k)} u_j v_j))$, hence

$$\begin{aligned} K_1(X_{2l(k)-1}) &\simeq K_i(\operatorname{Spec} k[u,v]/(1-uv)) \\ &\simeq K_i(\operatorname{Spec} k[u,u^{-1}]) \simeq K_i(k) \oplus K_{i-1}(k). \end{aligned}$$

This Corollary is the result proved by Jouanolou [**5**].

During the preparation of this paper, the author has become aware of recent work by A. Suslin [**11**, Theorem 2.3] proving part (2) of the Corollary by similar methods.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

Current address: Department of Mathematics, Michigan State University, East Lansing, Michigan 48824