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A REPRESENTATION THEOREM FOR SEMILATTICES

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ABSTRACT. We prove that every semilattice $(L, \wedge)$ admits an embedding Q into the set $R(X)$ of all partial orders on some set X such that for all $a, b \in L$, $Q(a \wedge b) = Q(a) \cap Q(b)$ and if $a \vee b$ exists then also $Q(a \vee b) = Q(b) \circ Q(a)$.

Denote by $R(X)$ the set of all partial orders on X. The set $R(X)$ is closed under the operation of intersection of binary relations. Therefore we can consider $R(X)$ as a semilattice $(R(X), \cap)$, which we call the semilattice of partial orders on X. The relation product $S \circ H$ of two partial orders $H, S \in R(X)$ need not belong to $R(X)$. But if it does then $S \circ H$ is the least upper bound of $\{H, S\}$ in $R(X)$ under the set-theoretical inclusion.

By a representation of a semilattice $(L, \wedge)$ we mean an embedding $F: L \to R(X)$ for some set X such that $F(a \wedge b) = F(a) \cap F(b)$ for all $a, b \in L$. The set X is called the domain of the representation F.

The main result of this note is the following theorem.

THEOREM. *Every semilattice $(L, \wedge)$ admits a representation Q such that for all $a, b \in L$, if $a \vee b$ exists then $Q(a \vee b) = Q(b) \circ Q(a)$.*

The proof of this theorem uses some techniques of B. Jónsson [2]. Let $(L, \wedge)$ be an arbitrary semilattice.

LEMMA 1. *$(L, \wedge)$ has a representation.*

Consider the mapping $F: L \to R(L)$ defined by

$$F(a) = \{(x, y) \in L \times L \mid x \leq y \text{ and } x, y \leq a\} \cup \Delta_L,$$

where Δ_L denotes the identity relation on L. It is easy to see that F is a representation of the semilattice $(L, \wedge)$.

We shall say that the representation P with domain Y is an extension of a representation F with domain X if $X \subset Y$ and $F(a) = P(a) \cap X \times X$ for all $a \in L$.

LEMMA 2. *If F is a representation of $(L, \wedge)$, $a \vee b$ exists, and $(p, q) \in F(a \vee b)$ where $p \neq q$, then there exists an extension P of F such that $(p, q) \in P(b) \circ P(a)$.*

Let $Y = X \cup \{r\}$ where X is the domain of representation F and r does not belong to X. Let x, y denote arbitrary elements of X, and define P by putting

$$(1) \qquad (x, y) \in P(c) \Leftrightarrow (x, y) \in F(c),$$

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(2) $$(x, r) \in P(c) \Leftrightarrow (x, p) \in F(c) \text{ and } a \leqslant c,$$

(3) $$(r, y) \in P(c) \Leftrightarrow (q, y) \in F(c) \text{ and } b \leqslant c,$$

(4) $$(r, r) \in P(c).$$

It is easy to see that $P(c) \in R(Y)$ for all $c \in L$. From (1) we see that $P(c) = F(c) \cap X \times X$, and the fact that F is one-to-one implies the same for P. Using (1)–(4) it is easily seen that $P(c \wedge d) = P(c) \cap P(d)$ for all $c, d \in L$. Therefore P is a representation. Since $(p, r) \in P(a)$ and $(r, q) \in P(b)$, we have $(p, q) \in P(b) \circ P(a)$, which completes the proof of Lemma 2.

Suppose that α is an inaccessible cardinal such that $|L| < \alpha$ where $|L|$ is the power of the set L. Denote by Σ the set of all representations F: $L \to R(X)$ such that $|X| < \alpha$. By Lemma 1, $\Sigma \neq \varnothing$. Suppose that $\{F_i \mid i \in I\}$ is a family of representations belonging to Σ such that the family $\{X_i \mid i \in I\}$ of the corresponding domains is a chain under set-theoretical inclusion and F_j is an extension of F_i if $X_i \subset X_j$. Then it is easy to see that the mapping F: $L \to R(X)$ where $X = \bigcup\{X_i \mid i \in I\}$ and $F(c) = \bigcup\{F_i(c) \mid i \in I\}$ for all $c \in L$, is a representation belonging to Σ. Hence by the Zorn Lemma there exists a representation Q which is maximal in the sense that if P is an extension of Q then P equals Q.

Let $a, b \in L$ and suppose that $a \vee b$ exists. First we show that $Q(a \vee b) \subset Q(b) \circ Q(a)$. Suppose the contrary, i.e. that there exist x, y such that $(x, y) \in Q(a \vee b)$ and $(x, y) \notin Q(b) \circ Q(a)$. Then, by Lemma 2, there exists an extension P of Q such that $(x, y) \in P(b) \circ P(a)$, but this contradicts the maximality of Q. Thus $Q(a \vee b) \subset Q(b) \circ Q(a)$. Since $a \leqslant a \vee b$ and $b \leqslant a \vee b$, we have $Q(a) \subset Q(a \vee b)$ and $Q(b) \subset Q(a \vee b)$. Therefore $Q(b) \circ Q(a) \subset Q(a \vee b) \circ Q(a \vee b) \subset Q(a \vee b)$. Thus $Q(a \vee b) = Q(b) \circ Q(a)$, and this completes the proof of the theorem.

From this theorem the following result of **[1, 3]** is obtained immediately.

COROLLARY. *Every lattice* $(L, \wedge, \vee)$ *is isomorphic to an algebra of binary relations* $(\mathcal{P}, \cap, \circ)$, *where all elements of* $\mathcal{P}$ *are partial orders.*

Note that the construction used in the proof of our theorem is simpler than the one in **[1, 3]**.

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