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# THE LEAST AREA BOUNDED BY MULTIPLES OF A CURVE

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ABSTRACT. For each positive integer $n$, we construct a smooth curve $\Gamma$ in $\mathbf{R}^4$ such that the least area of a surface (integral current) with boundary $n\Gamma$ is less than $n/k$ of the least area of a surface with boundary $k\Gamma$ $(1 \leqslant k < n)$.

If $T$ is an area-minimizing surface (integral current), must $2T$ also be area-minimizing? Or is it possible that there is some other surface $S$ such that $\partial S = 2 \cdot \partial T$ and $\text{Area}(S) < 2 \cdot \text{Area}(T)$? In 1968, L. C. Young showed that it is possible [**Y**]. That is, he showed that if we denote by $\alpha(\Gamma)$ the least area of any surface with boundary $\Gamma$, then there is a curve $\Gamma$ in $\mathbf{R}^4$ such that $\alpha(2\Gamma) < 2\alpha(\Gamma)$. Here we generalize Young's example by constructing, for each $n > 1$, a smooth simple closed curve $\Gamma$ in $\mathbf{R}^4$ such that $(1/n)\alpha(n\Gamma) < (1/k)\alpha(k\Gamma)$ for $1 \leqslant k < n$. Moreover, if for each $n$ we let $\Gamma_n$ be such a curve with $\text{length}(\Gamma_n) = 2^{-n}$, then the sum $\Gamma$ (over $n$) of suitably spaced translates of $\Gamma_n$ is such that $\inf\{k^{-1}\alpha(k\Gamma): k = 1, 2, 3, \ldots\}$ is not attained for any $k$. (In [**F**, 2.8] it is shown that this infimum is equal to the least possible area of a *real* flat chain with boundary $\Gamma$.) Finally, one can, by adding suitably narrow bridges, even make $\Gamma$ into a single connected curve.

A completely different example of this phenomenon has been discovered by Frank Morgan [**M**]. His proof uses symmetries of the curve he constructs.

**Preliminaries.** 1. If $\sigma \subset \mathbf{R}^4$ is a line segment, $C(\sigma, \varepsilon)$ is the solid cylinder of radius $\varepsilon$ about $\sigma$:

$$C(\sigma, \varepsilon) = \{x + y : x \in \sigma, |y| < \varepsilon, y \text{ is perpendicular to } \sigma\}.$$

One can show (using, for example, the coarea formula) that if $T$ is a surface (integral current) in $\mathbf{R}^4$ with

$$(\partial T) \llcorner C(\sigma, \varepsilon) = k\sigma,$$

then

$$\text{Area}(T \llcorner C(\sigma, \varepsilon)) \geqslant k\varepsilon|\sigma|,$$

where $|\sigma|$ is the length of $\sigma$.

2. Suppose $T$ is an area-minimizing surface that passes through a point $p$ such that $\mathbf{B}(p, r) \cap \text{spt}\,\partial T = \varnothing$. Then

$$\text{Area}(T \llcorner \mathbf{B}(p, r)) \geqslant \pi r^2.$$

(Use the monotonicity theorem for minimal surfaces.)

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**The construction.** Consider the topological space obtained by identifying the edges of an $n$-gon with each other. Let $K$ be a piecewise linearly embedded image of it in $\mathbf{R}^4$. (The mapping $f(z) = \langle(1 - |z|^2)z, z^n\rangle$ from $\mathbf{B}^2(0, 1) \subset \mathbf{R}^2 = \mathbf{C}$ to $\mathbf{R}^4 = \mathbf{C} \times \mathbf{C}$ is a continuous embedding of it.) Let $\delta > 0$ be such that $K(2\delta) = \{x \in \mathbf{R}^4: \operatorname{dist}(x, K) \leqslant 2\delta\}$ retracts onto $K$.

Note that for each positive $\varepsilon < \delta$, we can find segments $\sigma_i$ in $K$ such that the cylinders $C(\sigma_i, \varepsilon)$ are disjoint and cover all but $O(\varepsilon)$ of the area of $K$ (fill each polygonal face of $K$ with parallel segments spaced $2\varepsilon$ apart):

$$\text{(1)} \qquad \operatorname{Area}\left(K \cap \left(\bigcup_i C(\sigma_i, \varepsilon)\right)\right) \geqslant \operatorname{Area}(K) - O(\varepsilon).$$

Now join the ends of the segments $\sigma_i$ to get a curve $\Gamma$ in $K$ that is a generator for the first homology group $H_1(K) = Z_n$. (In other words, $\Gamma$ is a generator of $H_1(K)$ such that $\Gamma \cap C(\sigma_i, \varepsilon) = \sigma_i$ for each $i$.)

Now let $T$ be an area-minimizing surface with $\partial T = k\Gamma$ (where $1 \leqslant k < n$). Since $k\Gamma$ does not bound in $K$, it does not bound in $K(2\delta)$ (which retracts onto $K$). Thus $T$ must contain some point $p$ not in $K(2\delta)$. Now

$$\operatorname{Area}(T) \geqslant \sum_i \operatorname{Area}(T \cap C(\sigma_i, \varepsilon)) + \operatorname{Area}(T \cap \mathbf{B}(p, \delta))$$

$$\geqslant \sum_i k|\sigma_i|\varepsilon + \pi\delta^2$$

(see preliminaries)

$$\text{(2)} \qquad \geqslant \tfrac{1}{2}k[\operatorname{Area}(K) - O(\varepsilon)] + \pi\delta^2$$

(by (1)).

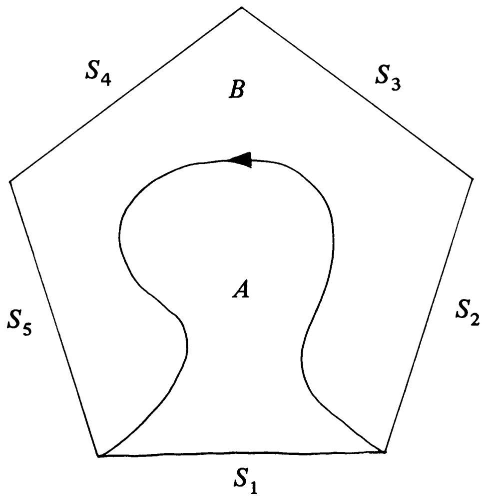


FIGURE 1. $\partial(4A - B) = (4\Gamma + 4S_1) - (-\Gamma + S_2 + S_3 + S_4 + S_5) = 5\Gamma$

On the other hand, $n\Gamma$ is a homologous to 0 in $K$. Since $H_2(K) = 0$, $n\Gamma$ bounds a unique 2-chain $S$ in $K$. $S$ consists of one region $A$ in $K$ taken with multiplicity $(n-1)$ together with the complementary region $B$ taken with multiplicity $-1$ (see Figure 1 for the case $n = 5$). Note that each region contains approximately (within $O(\varepsilon)$) half the area of $K$. (That is because $\Gamma$ divides each rectangle $K \cap C(\sigma_i, \varepsilon)$ into two equal pieces, and because these rectangles fill up most of $K$.) Thus

$$\begin{aligned} \text{Area}(S) &= \text{Area}(B) + (n-1)\,\text{Area}(A) \\ &\leqslant \tfrac{1}{2}\,\text{Area}(K) + \tfrac{1}{2}(n-1)\,\text{Area}(K) + O(\varepsilon) \\ &\leqslant \tfrac{1}{2}n\,\text{Area}(K) + O(\varepsilon). \end{aligned}$$

$$\therefore (1/n)\alpha(n\Gamma) \leqslant \tfrac{1}{2}\,\text{Area}(K) + O(\varepsilon). \tag{3}$$

Combining (2) and (3), we have for small enough $\varepsilon$,

$$(1/n)\alpha(n\Gamma) < (1/k)\alpha(k\Gamma) \quad \text{for } k = 1, 2, \ldots, n-1.$$

**Open questions.** 1. Is there a curve $\Gamma$ such that $\alpha(2\Gamma) < \alpha(\Gamma)$?

2. If so, is the ratio of $\alpha(2\Gamma)$ to $\alpha(\Gamma)$ bounded away from 0?

3. If so, is the ratio of $\inf\{(1/k)\alpha(k\Gamma)\colon k \in Z\}$ to $\alpha(\Gamma)$ bounded from 0?

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