PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 2, February 1984

# ON THE BOREL $\sigma$-FIELD AND THE BAIRE PROPERTY

JOHN C. MORGAN II

ABSTRACT. There is no category base for which the $\sigma$-field of sets with the Baire property coincides with the $\sigma$-field of linear Borel sets.

The sets having the classical Baire property with respect to a given topology being closed under the set-theoretical operation $\mathcal{A}$, it follows that the $\sigma$-field of Borel subsets of the real line $\mathbf{R}$ does not coincide with the $\sigma$-field of sets with the Baire property for any topology. Generalizing the notion of a topology is the notion of a category base, for which is is not known whether the family of sets having the Baire property is closed under operation $\mathcal{A}$. Nevertheless, we shall show that the result concerning the Borel $\sigma$-field remains valid for any category base. In particular, this shows that the notion of a category base is not so general that any $\sigma$-field is obtainable as the $\sigma$-field of sets with the Baire property for some category base. In establishing this result we shall utilize terminology and properties found in [**1**–**3**].

THEOREM. *There exists no category base* $(\mathbf{R}, \mathcal{C})$ *for which* $\mathfrak{B}(\mathcal{C})$ *coincides with the* $\sigma$*-field of all Borel sets in* $\mathbf{R}$.

PROOF. Under the assumption that there does exist such a category base, we establish the following seven propositions (the first and last of which are contradictory).

PROPOSITION 1. *Every meager set is countable.*

PROOF. If there were an uncountable meager set $S$ then, being a Borel set, $S$ has power $2^{\aleph_0}$. Every subset of $S$ has the Baire property and, accordingly, is a Borel set. We then obtain the contradictory existence of $2^{2^{\aleph_0}}$ different Borel sets.

PROPOSITION 2. *For any uncountable region* $A$, *the category base* $(A, \mathcal{C}_A)$ *does not satisfy CCC.*

PROOF. If $(A, \mathcal{C}_A)$ did satisfy CCC, then $\mathfrak{B}(\mathcal{C}_A)$, which coincides with the $\sigma$-field of all Borel subsets of $A$, would be closed under the set-theoretical operation $\mathcal{A}$ (cf. [**1**, Theorem 10]). Consequently, every analytic subset of $A$ would be a Borel set, contradicting the fact that every uncountable Borel set contains an analytic set which is not a Borel set.

Received by the editors May 4, 1983 and, in revised form, July 11, 1983.
1980 *Mathematics Subject Classification*. Primary 26A21, 28A05, 54A05; Secondary 04A15, 54E52.
*Key words and phrases*. Borel sets, category base, Baire category, Baire property.

PROPOSITION 3. *Every region contains a region which is of the first category.*

PROOF. Assume there is an uncountable region $A$ every subregion of which is of the second category. Because the family of all sets of the second category which have the classical Baire property contains all Borel sets of the second category and satisfies CCC, the category base $(A, \mathcal{C}_A)$ satisfies CCC, in contradiction to Proposition 2.

PROPOSITION 4. *There exists an uncountable family $\mathfrak{M}$ of disjoint regions satisfying the condition*

$$(*) \qquad \text{For every region } A \in \mathcal{C} \text{ there exists a region } M \in \mathfrak{M} \text{ such that } A \cap M \text{ contains a region in } \mathcal{C}.$$

PROOF. Let $\mathfrak{N}$ denote the family of all regions of the first category. Apply Proposition 3 and the basic Lemmas 1, 2 of [3] to obtain a maximal family $\mathfrak{M}$ of disjoint regions in $\mathfrak{N}$ satisfying the condition $(*)$ such that $\mathbf{R} - \bigcup \mathfrak{M}$ is a singular set. Every singular set being countable by Proposition 1, it follows from the Baire Category Theorem that the family $\mathfrak{M}$ is uncountable.

PROPOSITION 5. *At most countably many regions consist of a single point.*

PROOF. Assume there are uncountably many regions each consisting of a single point. The union $U$ of these regions is an uncountable set, every subset of which has the Baire property. For, if $T \subset U$ and $A$ is any region for which $A \cap T$ is abundant, then there is a point $x \in A \cap T$; whence, $B = \{x\}$ is a subregion of $A$ such that $B \cap (\mathbf{R} - T)$ is a meager set. Now, every subset of $U$ having the Baire property implies the contradictory existence of $2^{2^{\aleph_0}}$ Borel sets.

PROPOSITION 6. *There are at most countably many disjoint regions which contain no point $x$ for which $\{x\}$ is a meager set.*

PROOF. Let $A$ be such a region and let $p$ be any point of $A$. From the Fundamental Theorem it follows that there exists a region $B \subset A$ in which $\{p\}$ is abundant everywhere. Since $\{p\}$, being Borel, has the Baire property, the set $B \cap (\mathbf{R} - \{p\})$ is meager. Because $\{x\}$ is abundant for every $x \in A$, we must have $B \cap (\mathbf{R} - \{p\}) \neq \varnothing$, so that $B = \{p\}$. Every such region $A$ thus contains a region $B$ consisting of a single point. The assertion is now a consequence of Proposition 5.

PROPOSITION 7. *There exists an uncountable meager set.*

PROOF. Let $\mathfrak{M}$ denote the family of regions given in Proposition 4. According to Proposition 6, there are at most countably many regions in $\mathfrak{M}$ which contain no point $x$ for which $\{x\}$ is a meager set. Delete all such regions from $\mathfrak{M}$ and let $\mathfrak{M}'$ denote the uncountable family remaining. In each region $M \in \mathfrak{M}'$ select one point $x$ for which $\{x\}$ is a meager set and let $S$ denote the uncountable set of points selected. Using the condition $(*)$ it is seen that every region $A \in \mathcal{C}$ contains a region $B \in \mathcal{C}$ such that $S \cap B$ is a meager set. By the Fundamental Theorem, $S$ is a meager set.

REMARK. The statement of Proposition 4 holds for the category base of all nowhere dense perfect sets. Choosing one point in each region $M \in \mathfrak{M}$, one then obtains an uncountable Marczewski singular set; i.e. an uncountable set having the property $(s_2^0)$ of [**4**].

## REFERENCES

1. J. C. Morgan, II. *Baire category from an abstract viewpoint*, Fund. Math. **94** (1977), 13–23.

2. ______, *On product bases*, Pacific J. Math. **99** (1982), 105–126.

3. ______, *Measurability and the abstract Baire property* (to appear).

4. E. Szpilrajn (Marczewski), *Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles*, Fund. Math. **24** (1935), 17–34.

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE POLYTECHNIC UNIVERSITY, POMONA, CALIFORNIA 91768