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A DESCRIPTION OF WEIGHTS SATISFYING THE A_∞ CONDITION OF MUCKENHOUPT

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ABSTRACT. A nonnegative weight w on R^n satisfies the A_∞ condition iff

$$\sup_{Q\in\mathcal{Q}}\left(|Q|^{-1}\cdot\int_Q w\,dx\right)\cdot\exp\left\{\frac{1}{|Q|}\int_Q\log\frac{1}{w}\,dx\right\}<\infty.$$

Here $\mathcal{Q}$ stands for a family of all cubes in R^n. Applications to BMO are considered.

A nonnegative weight w defined on R^n satisfies the A_p condition (briefly $w\in A_p(R^n)$), $1<p<\infty$, if

$$\left(\frac{1}{|Q|}\int_Q w\,dx\right)\cdot\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\,dx\right)^{p-1}<C, \tag{1}$$

where C is independent of Q, Q ranges over the family $\mathcal{Q}$ of all cubes in R^n, and $|Q|$ is the Lebesgue measure of Q.

The class $A_1(R^n)$ is defined as a class of all w satisfying

$$\frac{1}{|Q|}\int_Q w\,dx\leqslant C\cdot\operatorname*{ess-inf}_Q w. \tag{2}$$

The class $A_\infty(R^n)$ has been introduced by C. Fefferman for $p=\infty$. It is a class of all weights w for which there exist positive numbers α, δ in $(0,1)$ such that for every pair (F,Q), F being a measurable subset of $Q\in\mathcal{Q}$ satisfying $|F|<\alpha|Q|$, the following inequality holds:

$$\int_F w\,dx<\delta\cdot\int_Q w\,dx. \tag{3}$$

B. Muckenhoupt [1] has shown that $A_\infty(R^n)=\bigcup_{p<\infty}A_p(R^n)$ (see [2] for a simplification).

It is clear that (2) can be considered as a limit case of (1) because

$$\lim_{p\to 1+0}\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\,dx\right)^{p-1}=\frac{1}{\operatorname{ess-inf}_Q w}.$$

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This idea applied to the case $p = \infty$ leads to the following condition on w:

$$C(w) = \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q w\,dx \right) \cdot \exp\left\{ \frac{1}{|Q|} \int_Q \log \frac{1}{w}\,dx \right\} < +\infty. \tag{4}$$

Indeed

$$\lim_{p \to +\infty} \left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)}\,dx \right)^{p-1} = \lim_{r \to 0+} \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{w} \right)^r dx \right)^{1/r}$$
$$= \exp\left\{ \frac{1}{|Q|} \int_Q \log \frac{1}{w}\,dx \right\}$$

(see [3, Chapter IV, §6, Ex. 6-c for example]).

Let $B_\infty(R^n)$ denote the class of all weights satisfying (4). Jensen's inequality

$$\exp\left\{ \frac{1}{|Q|} \int_Q \log \frac{1}{w}\,dx \right\} \leqslant \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{w} \right)^{1/(p-1)} dx \right)^{p-1},$$

together with Muckenhoupt's result [1], imply $A_\infty(R^n) \subset B_\infty(R^n)$.

THEOREM 1. $A_\infty(R^n) = B_\infty(R^n)$.

PROOF. Let $w \in B_\infty(R^n)$ and let $C(w)$ denote the left-hand side of (4). Consider (F, Q) with $0 < |F| < |Q|$ and let $E = Q \setminus F$. Then

$$\int_E w\,dx = \varepsilon \cdot \int_F w\,dx, \tag{5}$$

where $\varepsilon > 0$. For brevity set $m_A(f) \stackrel{\text{def}}{=} (1/|A|) \int_A f\,dx$. Clearly,

$$\log C(w) \geqslant m_Q\left(\log \frac{1}{w} \right) + \log m_Q(w) \tag{6}$$
$$= \frac{|E|}{|Q|} m_E\left(\log \frac{1}{w} \right) + \frac{|F|}{|Q|} \cdot m_F\left(\log \frac{1}{w} \right) + \log m_Q(w).$$

Jensen's inequality

$$m_A\left(\log \frac{1}{w} \right) = -m_A(\log w) \geqslant -\log m_A(w)$$

applied for $A = E, F$ yields

$$\log C(w) \geqslant -\frac{|E|}{|Q|} \log m_E(w) - \frac{|F|}{|Q|} \log m_F(w) + \log m_Q(w).$$

It follows from (5) that

$$\log C(w) \geqslant -\frac{|E|}{|Q|} \log m_E(w) - \frac{|F|}{|Q|} \log m_E(w) + \frac{|F|}{|Q|} \log \frac{|F|}{|E|}$$
$$+ \frac{|F|}{|Q|} \log \varepsilon + \log m_Q(w)$$
$$= \frac{|F|}{|Q|} \log \frac{|F|}{|E|} + \log \frac{|E|}{|Q|} + \frac{|F|}{|Q|} \log \varepsilon + \log\left(1 + \frac{1}{\varepsilon} \right)$$
$$= \frac{|F|}{|Q|} \log \frac{|F|}{|Q|} + \frac{|E|}{|Q|} \log \frac{|E|}{|Q|} + \log(1 + \varepsilon) + \frac{|E|}{|Q|} \log \frac{1}{\varepsilon}.$$

Now let $\varphi(x) = x\log x + (1-x)\log(1-x)$. Clearly $\inf_{0<x<1}\varphi(x) = -\log 2$. This implies

$$\frac{|E|}{|Q|}\log\frac{1}{\varepsilon} \leqslant \log 2C(w). \tag{7}$$

Suppose $|F| < \frac{1}{2}|Q|$. It follows from (7) that either $\varepsilon \geqslant 1$ or $\varepsilon \geqslant (4C^2(w))^{-1}$. Therefore

$$\int_F w\,dx = \frac{1}{1+\varepsilon}\int_Q w\,dx \leqslant \delta\cdot\int_Q w\,dx$$

with $\delta = \max(1/2, 4C^2(w)/(1+4C^2(w)))$. □

It is well known that $w \in A_\infty(R^n) \Rightarrow \log w \in \mathrm{BMO}$. Theorem 1 implies therefore $\log(B_\infty(R^n)) \subset \mathrm{BMO}$. This can be reformulated as follows.

COROLLARY. *Let $f \in L^1_{\mathrm{loc}}(R^n)$ and suppose there exists a positive constant C such that*

$$\log m_Q(e^f) \leqslant m_Q(f) + C \tag{8}$$

for $Q \in \mathcal{Q}$. Then $f \in$ BMO.

For the proof it is sufficient to remark that (8) is equivalent to (4) with $w = \exp(f)$ and then apply Theorem 1.

This fact can also be proved directly, which leads to an alternative description of $A_\infty(R^n)$ similar to that obtained in [4] for $A_1(R^n)$. Let $y^+ = \max(y, 0)$.

LEMMA 1. *Let $w \in B_\infty(R^n)$. Then*

$$\sup_{Q\in\mathcal{Q}}\frac{1}{|Q|}\int_Q \log^+\frac{m_Q(w)}{w}\,dx \leqslant C(w) + \frac{1}{e}.$$

PROOF. Fix Q in $\mathcal{Q}$ and consider $F = \{x \in Q\colon w(x) < m_Q(w)\}$. Without loss of generality $|F| > 0$, for otherwise $w = m_Q(w)$ a.e. on Q. Clearly $|F| < |Q|$ and it is possible to define $\varepsilon > 0$ by (5). As above (see (6)),

$$\begin{aligned}
\log C(w) &\geqslant m_Q\left(\log\frac{1}{w}\right) + \log m_Q(w)\\
&\geqslant -\frac{|E|}{|Q|}\log m_E(w) + \frac{|F|}{|Q|}m_F\left(\log\frac{1}{w}\right) + \log m_Q(w)\\
&= \frac{1}{|Q|}\int_F \log\frac{m_Q(w)}{w}\,dx + \frac{|E|}{|Q|}\log\frac{|E|}{|Q|} + \frac{|E|}{|Q|}\log\left(1+\frac{1}{\varepsilon}\right)\\
&\geqslant \frac{1}{|Q|}\int_Q \log^+\frac{m_Q(w)}{w}\,dx - \frac{1}{e}. \quad \square
\end{aligned}$$

For $w \geqslant 0$, Jensen's inequality yields $m_Q(\log w) \leqslant \log m_Q(w)$, which obviously implies that $m_Q(\log m_Q(w) - \log w) \geqslant 0$ and, therefore,

$$\frac{1}{|Q|}\int_Q |\log m_Q(w) - \log w|\,dx \leqslant \frac{2}{|Q|}\int_Q \log^+\frac{m_Q(w)}{w}\,dx. \tag{9}$$

Now define a space BLO^1 consisting of all functions f which have a bounded low oscillation in the L^1-metric

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|}\int_Q \left(\log m_Q(e^f) - f\right)^+ dx < +\infty.$$

It is clear that $\mathrm{BLO}^1 \subset \mathrm{BMO}$ (set $w = \exp f$ in (9)). Moreover, $f \in \mathrm{BMO}$ iff $\varepsilon f \in \mathrm{BLO}^1$ for some $\varepsilon > 0$, as follows from

THEOREM 2. *$w \in A_\infty(R^n)$ iff $\log w \in \mathrm{BLO}^1$.*

PROOF. If $w \in B_\infty(R^n)$, then $\log w \in \mathrm{BLO}^1$ by Lemma 1.
If $\log w \in \mathrm{BLO}^1$, then (9) implies

$$\begin{aligned}\log m_Q(w) - m_Q(\log w) &= \frac{1}{|Q|}\int_Q \left(\log m_Q(w) - \log w\right) dx \\ &\leqslant \frac{2}{|Q|}\int_Q \log^+ \frac{m_Q(w)}{w}\, dx,\end{aligned}$$

which obviously yields $w \in B_\infty(R^n)$. $\square$

In this connection it is reasonable to consider a space BLO^∞ of all functions having a bounded low oscillation in the L^∞-metric

$$\sup_{Q\in\mathcal{Q}} \left\{\log m_Q(e^f) - \inf_Q f\right\} < +\infty.$$

Clearly, $w \in A_1(R^n)$ iff $\log w \in \mathrm{BLO}^\infty$. Coifman and Rochberg have introduced **[4]** the space BLO consisting of all functions satisfying

$$\sup_{Q\in\mathcal{Q}} \left\{m_Q(f) - \inf_Q f\right\} < +\infty.$$

It has been noted in **[4]** that $w \in A_1(R^n)$ iff $\varepsilon \log w \in \mathrm{BLO}$ for some $\varepsilon > 0$.

It is easy to check that $\mathrm{BLO}^\infty = \mathrm{BLO}^1 \cap \mathrm{BLO}$. Indeed, $\mathrm{BLO}^\infty \subset \mathrm{BLO}^1$ and the assumption $f \in \mathrm{BLO}^1$ implies

$$0 \leqslant \log m_Q(e^f) - m_Q(f) \leqslant C < +\infty.$$

In conclusion, note that the equality $B_\infty(R^n) = \bigcup_{p<\infty} A_p(R^n)$ can easily be proved with the help of well-known facts from the theory of BMO. If $w \in B_\infty(R^n)$ then $\log w \in \mathrm{BMO}$. By the theorem of John and Nirenberg (see also (8)) there exist positive c_1, c_2 such that

$$\left|\left\{x \in Q\colon \left|\log m_Q(w) - \log w\right| > \lambda\right\}\right| < c_1 \cdot |Q| e^{-c_2\cdot\lambda}, \qquad \lambda > 0.$$

Let

$$E_k = \left\{x \in Q\colon e^k m_Q(w)^{-1} \leqslant w^{-1} \leqslant e^{k+1} m_Q(w)^{-1}\right\}, \qquad k = 0, 1, 2, \ldots.$$

Then

$$\frac{1}{|Q|}\int_Q\left(\frac{1}{w}\right)^{1/(p-1)}dx \leqslant m_Q(w)^{-1/(p-1)}\cdot\left\{1+\sum_{k\geqslant 0}\frac{|E_k|}{|Q|}e^{(k+1)/(p-1)}\right\}$$
$$\leqslant m_Q(w)^{-1/(p-1)}\left\{1+c_1\cdot\sum_{k\geqslant 0}\exp\left(\frac{k+1}{p-1}-c_2\cdot K\right)\right\}$$
$$\leqslant C\cdot m_Q(w)^{-1/(p-1)}$$

if $1/(p-1)<c_2$. Therefore $w\in A_p(R^n)$ if $p>1+1/c_2$. The inclusion $\bigcup_{p<\infty}A_p(R^n)\subset B_\infty(R^n)$ is a simple consequence of Jensen's inequality.

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