

A DESCRIPTION OF WEIGHTS SATISFYING THE A_∞ CONDITION OF MUCKENHOUP

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ABSTRACT. A nonnegative weight w on R^n satisfies the A_∞ condition iff

$$\sup_{Q \in \mathcal{Q}} \left(|Q|^{-1} \int_Q w \, dx \right) \cdot \exp \left\{ \frac{1}{|Q|} \int_Q \log \frac{1}{w} \, dx \right\} < \infty.$$

Here \mathcal{Q} stands for a family of all cubes in R^n . Applications to BMO are considered.

A nonnegative weight w defined on R^n satisfies the A_p condition (briefly $w \in A_p(R^n)$), $1 < p < \infty$, if

$$(1) \quad \left(\frac{1}{|Q|} \int_Q w \, dx \right) \cdot \left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)} \, dx \right)^{p-1} < C,$$

where C is independent of Q , Q ranges over the family \mathcal{Q} of all cubes in R^n , and $|Q|$ is the Lebesgue measure of Q .

The class $A_1(R^n)$ is defined as a class of all w satisfying

$$(2) \quad \frac{1}{|Q|} \int_Q w \, dx \leq C \cdot \operatorname{ess-inf}_Q w.$$

The class $A_\infty(R^n)$ has been introduced by C. Fefferman for $p = \infty$. It is a class of all weights w for which there exist positive numbers α, δ in $(0, 1)$ such that for every pair (F, Q) , F being a measurable subset of $Q \in \mathcal{Q}$ satisfying $|F| < \alpha|Q|$, the following inequality holds:

$$(3) \quad \int_F w \, dx < \delta \cdot \int_Q w \, dx.$$

B. Muckenhoupt [1] has shown that $A_\infty(R^n) = \bigcup_{p < \infty} A_p(R^n)$ (see [2] for a simplification).

It is clear that (2) can be considered as a limit case of (1) because

$$\lim_{p \rightarrow 1+0} \left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)} \, dx \right)^{p-1} = \frac{1}{\operatorname{ess-inf}_Q w}.$$

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This idea applied to the case $p = \infty$ leads to the following condition on w :

$$(4) \quad C(w) = \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q w \, dx \right) \cdot \exp \left\{ \frac{1}{|Q|} \int_Q \log \frac{1}{w} \, dx \right\} < +\infty.$$

Indeed

$$\begin{aligned} \lim_{p \rightarrow +\infty} \left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)} \, dx \right)^{p-1} &= \lim_{r \rightarrow 0+} \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{w} \right)^r \, dx \right)^{1/r} \\ &= \exp \left\{ \frac{1}{|Q|} \int_Q \log \frac{1}{w} \, dx \right\} \end{aligned}$$

(see [3, Chapter IV, §6, Ex. 6-c for example]).

Let $B_\infty(R^n)$ denote the class of all weights satisfying (4). Jensen's inequality

$$\exp \left\{ \frac{1}{|Q|} \int_Q \log \frac{1}{w} \, dx \right\} \leq \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{w} \right)^{1/(p-1)} \, dx \right)^{p-1},$$

together with Muckenhoupt's result [1], imply $A_\infty(R^n) \subset B_\infty(R^n)$.

THEOREM 1. $A_\infty(R^n) = B_\infty(R^n)$.

PROOF. Let $w \in B_\infty(R^n)$ and let $C(w)$ denote the left-hand side of (4). Consider (F, Q) with $0 < |F| < |Q|$ and let $E = Q \setminus F$. Then

$$(5) \quad \int_E w \, dx = \varepsilon \cdot \int_F w \, dx,$$

where $\varepsilon > 0$. For brevity set $m_A(f) \stackrel{\text{def}}{=} (1/|A|) \int_A f \, dx$. Clearly,

$$(6) \quad \begin{aligned} \log C(w) &\geq m_Q \left(\log \frac{1}{w} \right) + \log m_Q(w) \\ &= \frac{|E|}{|Q|} m_E \left(\log \frac{1}{w} \right) + \frac{|F|}{|Q|} \cdot m_F \left(\log \frac{1}{w} \right) + \log m_Q(w). \end{aligned}$$

Jensen's inequality

$$m_A \left(\log \frac{1}{w} \right) = -m_A(\log w) \geq -\log m_A(w)$$

applied for $A = E, F$ yields

$$\log C(w) \geq -\frac{|E|}{|Q|} \log m_E(w) - \frac{|F|}{|Q|} \log m_F(w) + \log m_Q(w).$$

It follows from (5) that

$$\begin{aligned} \log C(w) &\geq -\frac{|E|}{|Q|} \log m_E(w) - \frac{|F|}{|Q|} \log m_E(w) + \frac{|F|}{|Q|} \log \frac{|F|}{|E|} \\ &\quad + \frac{|F|}{|Q|} \log \varepsilon + \log m_Q(w) \\ &= \frac{|F|}{|Q|} \log \frac{|F|}{|E|} + \log \frac{|E|}{|Q|} + \frac{|F|}{|Q|} \log \varepsilon + \log \left(1 + \frac{1}{\varepsilon} \right) \\ &= \frac{|F|}{|Q|} \log \frac{|F|}{|Q|} + \frac{|E|}{|Q|} \log \frac{|E|}{|Q|} + \log(1 + \varepsilon) + \frac{|E|}{|Q|} \log \frac{1}{\varepsilon}. \end{aligned}$$

Now let $\varphi(x) = x \log x + (1 - x) \log(1 - x)$. Clearly $\inf_{0 < x < 1} \varphi(x) = -\log 2$. This implies

$$(7) \quad \frac{|E|}{|Q|} \log \frac{1}{\varepsilon} \leq \log 2C(w).$$

Suppose $|F| < \frac{1}{2}|Q|$. It follows from (7) that either $\varepsilon \geq 1$ or $\varepsilon \geq (4C^2(w))^{-1}$. Therefore

$$\int_F w \, dx = \frac{1}{1 + \varepsilon} \int_Q w \, dx \leq \delta \cdot \int_Q w \, dx$$

with $\delta = \max(1/2, 4C^2(w)/(1 + 4C^2(w)))$. \square

It is well known that $w \in A_\infty(R^n) \Rightarrow \log w \in \text{BMO}$. Theorem 1 implies therefore $\log(B_\infty(R^n)) \subset \text{BMO}$. This can be reformulated as follows.

COROLLARY. *Let $f \in L^1_{\text{loc}}(R^n)$ and suppose there exists a positive constant C such that*

$$(8) \quad \log m_Q(e^f) \leq m_Q(f) + C$$

for $Q \in \mathcal{Q}$. Then $f \in \text{BMO}$.

For the proof it is sufficient to remark that (8) is equivalent to (4) with $w = \exp(f)$ and then apply Theorem 1.

This fact can also be proved directly, which leads to an alternative description of $A_\infty(R^n)$ similar to that obtained in [4] for $A_1(R^n)$. Let $y^+ = \max(y, 0)$.

LEMMA 1. *Let $w \in B_\infty(R^n)$. Then*

$$\sup_{Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q \log^+ \frac{m_Q(w)}{w} \, dx \leq C(w) + \frac{1}{e}.$$

PROOF. Fix Q in \mathcal{Q} and consider $F = \{x \in Q: w(x) < m_Q(w)\}$. Without loss of generality $|F| > 0$, for otherwise $w = m_Q(w)$ a.e. on Q . Clearly $|F| < |Q|$ and it is possible to define $\varepsilon > 0$ by (5). As above (see (6)),

$$\begin{aligned} \log C(w) &\geq m_Q\left(\log \frac{1}{w}\right) + \log m_Q(w) \\ &\geq -\frac{|E|}{|Q|} \log m_E(w) + \frac{|F|}{|Q|} m_F\left(\log \frac{1}{w}\right) + \log m_Q(w) \\ &= \frac{1}{|Q|} \int_F \log \frac{m_Q(w)}{w} \, dx + \frac{|E|}{|Q|} \log \frac{|E|}{|Q|} + \frac{|E|}{|Q|} \log\left(1 + \frac{1}{\varepsilon}\right) \\ &\geq \frac{1}{|Q|} \int_Q \log^+ \frac{m_Q(w)}{w} \, dx - \frac{1}{e}. \quad \square \end{aligned}$$

For $w \geq 0$, Jensen's inequality yields $m_Q(\log w) \leq \log m_Q(w)$, which obviously implies that $m_Q(\log m_Q(w) - \log w) \geq 0$ and, therefore,

$$(9) \quad \frac{1}{|Q|} \int_Q |\log m_Q(w) - \log w| \, dx \leq \frac{2}{|Q|} \int_Q \log^+ \frac{m_Q(w)}{w} \, dx.$$

Now define a space BLO^1 consisting of all functions f which have a bounded low oscillation in the L^1 -metric

$$\sup_{Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q (\log m_Q(e^f) - f)^+ dx < +\infty.$$

It is clear that $BLO^1 \subset BMO$ (set $w = \exp f$ in (9)). Moreover, $f \in BMO$ iff $\varepsilon f \in BLO^1$ for some $\varepsilon > 0$, as follows from

THEOREM 2. $w \in A_\infty(R^n)$ iff $\log w \in BLO^1$.

PROOF. If $w \in B_\infty(R^n)$, then $\log w \in BLO^1$ by Lemma 1.

If $\log w \in BLO^1$, then (9) implies

$$\begin{aligned} \log m_Q(w) - m_Q(\log w) &= \frac{1}{|Q|} \int_Q (\log m_Q(w) - \log w) dx \\ &\leq \frac{2}{|Q|} \int_Q \log^+ \frac{m_Q(w)}{w} dx, \end{aligned}$$

which obviously yields $w \in B_\infty(R^n)$. \square

In this connection it is reasonable to consider a space BLO^∞ of all functions having a bounded low oscillation in the L^∞ -metric

$$\sup_{Q \in \mathcal{Q}} \left\{ \log m_Q(e^f) - \inf_Q f \right\} < +\infty.$$

Clearly, $w \in A_1(R^n)$ iff $\log w \in BLO^\infty$. Coifman and Rochberg have introduced [4] the space BLO consisting of all functions satisfying

$$\sup_{Q \in \mathcal{Q}} \left\{ m_Q(f) - \inf_Q f \right\} < +\infty.$$

It has been noted in [4] that $w \in A_1(R^n)$ iff $\varepsilon \log w \in BLO$ for some $\varepsilon > 0$.

It is easy to check that $BLO^\infty = BLO^1 \cap BLO$. Indeed, $BLO^\infty \subset BLO^1$ and the assumption $f \in BLO^1$ implies

$$0 \leq \log m_Q(e^f) - m_Q(f) \leq C < +\infty.$$

In conclusion, note that the equality $B_\infty(R^n) = \bigcup_{p < \infty} A_p(R^n)$ can easily be proved with the help of well-known facts from the theory of BMO . If $w \in B_\infty(R^n)$ then $\log w \in BMO$. By the theorem of John and Nirenberg (see also (8)) there exist positive c_1, c_2 such that

$$\left| \left\{ x \in Q : \left| \log m_Q(w) - \log w \right| > \lambda \right\} \right| < c_1 \cdot |Q| e^{-c_2 \lambda}, \quad \lambda > 0.$$

Let

$$E_k = \left\{ x \in Q : e^k m_Q(w)^{-1} \leq w^{-1} \leq e^{k+1} m_Q(w)^{-1} \right\}, \quad k = 0, 1, 2, \dots$$

Then

$$\begin{aligned} \frac{1}{|Q|} \int_Q \left(\frac{1}{w} \right)^{1/(p-1)} dx &\leq m_Q(w)^{-1/(p-1)} \cdot \left\{ 1 + \sum_{k \geq 0} \frac{|E_k|}{|Q|} e^{(k+1)/(p-1)} \right\} \\ &\leq m_Q(w)^{-1/(p-1)} \left\{ 1 + c_1 \cdot \sum_{k \geq 0} \exp\left(\frac{k+1}{p-1} - c_2 \cdot K \right) \right\} \\ &\leq C \cdot m_Q(w)^{-1/(p-1)} \end{aligned}$$

if $1/(p-1) < c_2$. Therefore $w \in A_p(\mathbb{R}^n)$ if $p > 1 + 1/c_2$. The inclusion $\bigcup_{p < \infty} A_p(\mathbb{R}^n) \subset B_\infty(\mathbb{R}^n)$ is a simple consequence of Jensen's inequality.

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