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PROBABILITY MEASURE REPRESENTATION OF NORMS ASSOCIATED WITH THE NOTION OF ENTROPY

ROMUALD DABROWSKI

ABSTRACT. One of the applications of Banach spaces introduced by B. Korenblum [1,2] is a new convergence test [2] for Fourier series including both Dirichlet-Jordan and the Dini-Lipschitz tests [3]. The norms of the spaces are given in terms of κ-entropy where $\kappa(s) \geqslant 0$, $0 < s \leqslant 1$, is a nondecreasing concave function such that $\kappa(1) = 1$. The κ-norms fill the gap between the uniform and the variation norms. The original proof of the general properties of κ-norms uses both combinatorial and approximation arguments which are rather complicated. We give a simple proof introducing a probabilistic representation of the norms so that the κ-norm of a real function f on $T = R/2\pi Z$ is the expectation of the mean oscillation of f on a subinterval of T, chosen in a suitable random process.

The κ-entropy. We start with the definition of the κ-norm introduced by B. Korenblum [1,2]. Let $T = R/2\pi Z$, and let $|E| = \int_E dx$ denote the normalized Lebesgue measure of a Borel subset E of T. The distance between points $x, y \in T$ is $d(x, y) = \min\{|x - y + 2\pi n|;\ n \in Z\}/2\pi$. $L^\infty(T)$ is the space of complex valued essentially bounded function on T, and $RL^\infty(T)$ denotes the space of real-valued function in $L^\infty(T)$. The graph of a function $f \in RL^\infty(T)$ is the set

$$\Gamma(f) = \Big\{(t, y) \in T \times R;\ \lim_{\delta \to 0}\left(\operatorname{ess\,inf}\{f(\tau);\ d(t, \tau) < \delta\}\right) \leqslant y \leqslant \lim_{\delta \to 0}\left(\operatorname{ess\,sup}\{f(\tau);\ d(t, \tau) < \delta\}\right)\Big\}.$$

One shows that $\Gamma(f)$ is always a closed subset of $T \times R$ which is connected on any subinterval of T.

DEFINITION 1. Let $\kappa(s)$, $0 < s \leqslant 1$, be a positive nondecreasing concave function such that $\kappa(1) = 1$. The *κ-entropy* of a finite subset E of T ($E \neq \varnothing$) is $\kappa(E) = \sum_{i=1}^n \kappa(|I_i|)$ where $\{I_i\}_{i=1}^n$ are the complementary intervals of E. For an infinite closed subset E of T ($E = \varnothing$), we set $\kappa(E) = \sup\{\kappa(F);\ F \subset E,\ F\text{-finite}\}$. We also put $\kappa(\varnothing) = 0$.

DEFINITION 2. For any function $f \in RL^\infty(T)$ we set

$$\|f\|_\kappa = \int_{-\infty}^{\infty} \kappa\big(\Gamma_y(f)\big)\, dy$$

where $\Gamma_y(f) = \{t;\ (t, y) \in \Gamma(f)\}$ is the y-level set of the graph of f.

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Notice that for a continuous real function f on T,

$$\|f\|_\kappa = \|f\|_C = \max\{f(t);\, t \in T\} - \min\{f(t);\, t \in T\} \quad \text{if } \kappa(s) = s,$$

and

$$\|f\|_\kappa = \|f\|_V = \sup\left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|\,;\, t_0 = 0 < t_1 < \cdots < t_n = 2\pi \right\} \quad \text{if } \kappa(s) = 1.$$

For an arbitrary function κ as in Definition 1 we have $s \leqslant \kappa(s) \leqslant 1$ for $0 < s \leqslant 1$ and, consequently, $\|\cdot\|_C \leqslant \|\cdot\|_\kappa \leqslant \|\cdot\|_V$. If $\kappa(0^+) = \lim_{s\to 0}(s) = \alpha > 0$, then $\kappa = \alpha + \beta\kappa_0$ where $\alpha + \beta = 1$ and $\kappa_0(0^+) = 0$. In this case $\|\cdot\|_\kappa = \alpha\|\cdot\|_V + \beta\|\cdot\|_{\kappa_0}$.

If $\kappa(0^+) = 0$ and $\lim_{s\to 0}(\kappa(s)/s) = A < \infty$, then $\|\cdot\|_C \leqslant \|\cdot\|_\kappa \leqslant A\|\cdot\|_C$, i.e. $\|\cdot\|_\kappa$ is equivalent to the norm $\|\cdot\|_C$. We, therefore, assume in what follows that $\kappa(0^+) = 0$ and $\lim_{s\to 0}(\kappa(s)/s) = \infty$. In this case $\|f\|_\kappa = \int_{T\times R}\kappa'(2d(\Gamma_y(f), x))\,dx\,dy$ where $\kappa(s) = \int_0^s \kappa'(t)\,dt$. It is by no means simple to show that $\|\cdot\|_\kappa$ is, in fact, a norm using just Definition 2 (the triangle inequality is hard to establish). We give another representation of κ-norms so that many properties of these norms follow naturally.

Probability measure representation of κ-norms. For closed subsets A, B of $T \times R$ we set $\lambda A = \{(t, \lambda y); (t, y) \in A\}$, $A + B = \{(t, y + y'); (t, y) \in A, (t, y') \in B\}$ and $AB = \{(t, yy'), (t, y) \in A, (t, y') \in B\}$. For $t \in T$ and $0 \leqslant s \leqslant \frac{1}{2}$ we define the oscillation of A on the interval $\{\tau;\, d(t, \tau) \leqslant s\}$ by the formula

$$\Omega_A(t, s) = \max\{y; (\tau, y) \in A, d(t, \tau) \leqslant s\} - \min\{y; (\tau, y) \in A, d(t, \tau) \leqslant s\}.$$

We also set $\|A\|_\infty = \max\{|y|, (t, y) \in A\}$. One easily verifies $\Omega_{\lambda A} = |\lambda|\Omega_A$ for any $\lambda \in R$, $\Omega_{A+B} \leqslant \Omega_A + \Omega_B$, $\Omega_A \leqslant \Omega_B$ for $A \subseteq B$ and $\Omega_{AB} \leqslant \|A\|_\infty\Omega_B + \|B\|_\infty\Omega_A$. For a function $f \in RL^\infty(T)$ we put $\Omega_f = \Omega_{\Gamma(f)}$. The following Lemma follows easily from the properties of Ω listed above.

LEMMA. *Let f, $g \in RL^\infty(T)$. We then have*

(i) $\Omega_{\lambda f} = |\lambda|\Omega_f$, $\lambda \in R$;

(ii) $\Omega_{f+g} \leqslant \Omega_f + \Omega_g$;

(iii) $\Omega_{fg} \leqslant \|f\|_\infty\Omega_g + \|g\|_\infty\Omega_f$;

(iv) *if f is continuously differentiable, then* $\Omega_f(t, s) \leqslant 2s\|f'\|_\infty$.

DEFINITION 3. Let μ be a probability measure on the unit interval $[0, 1]$. The μ-norm of a function $f \in RL^\infty(T)$ is

$$\|f\|_\mu = \int_{T\times[0,1]} \Omega_f(t, s/2)\,d\mu(s)\,dt,$$

where the value of $\Omega_f(t, s/2)/s$ at 0 is $\overline{\lim}_{s\to 0}(\Omega_f(t, s/2)/s)$. Notice that if μ is concentrated at 1, then $\|\cdot\|_\mu = \|\cdot\|_C$. If μ is concentrated at 0, and f is an absolutely continuous function on T, then $\|f\|_\mu = \|f\|_V$.

THEOREM 1. *Let* $\kappa(s)$, $0 < s \leqslant 1$, *be a positive nondecreasing concave function such that* $\kappa(0^+) = 0$ *and* $\lim_{s\to 0}(\kappa(s)/s) = \infty$. *There is a unique probability measure* μ_κ *on* $[0, 1]$ *such that*

$$\|f\|_\kappa = \|f\|_{\mu_\kappa} = \int_{T\times[0,1]} (\Omega_f/s)(t, s/2)\, dt\, d\mu_\kappa(s).$$

The map $\kappa \to \mu_\kappa$ *gives a one-to-one correspondence between the sets* $K_0 = \{\kappa;\ \kappa$ *described above*$\}$ *and* $P_0 = \{\mu;\ \mu$ *a probability measure on* $[0, 1]$ *such that* $\mu(\{0\}) = 0$, $\int_0^1 \mu(\tau)/\tau^2\, d\tau = \infty\}$.

REMARK. If μ is a probability measure on $[0, 1]$ such that $\mu(\{0\}) = 0$ then the condition $\int_0^1 \mu(\tau)/\tau^2\, d\tau = \infty$ is equivalent to the condition

$$\lim_{s\to 0} \frac{1}{s} \int_0^s \int_t^1 \frac{d\mu(\tau)}{\tau}\, dt = \infty.$$

For the proof of the theorem we will use this condition rather than the condition $\int_0^1 \mu(\tau)/\tau^2\, d\tau = \infty$.

PROOF. Let $\kappa \in K_0$, $\kappa'(s)$, $0 \leqslant s \leqslant 1$, is a nonnegative nonincreasing left-continuous function such that $\kappa'(0) = \infty$ and $\kappa(s) = \int_0^s \kappa'(t)\, dt$. We define $\alpha_\kappa(z) = \max\{s;\ \kappa'(s) \geqslant z\} = |\{s;\ \kappa'(s) \geqslant z\}|$. The function α_κ has the properties

(*) $$\alpha_\kappa(z) > 0 \quad \text{for all } z \geqslant 0,$$

(**) $$\kappa'(s) \geqslant z \quad \text{if and only if } s \leqslant \alpha_\kappa(z) \text{ for all } s \in [0, 1],\ z \geqslant 0,$$

(***) $$\kappa'(s) = \max\{z;\ \kappa'(s) \geqslant z\} = |\{z;\ \alpha_\kappa(z) \geqslant s\}|.$$

The measure μ_κ is defined by the formula $d\mu_\kappa(s) = s\,d\alpha_\kappa^{-1}(s)$ where $\int_E d\alpha_\kappa^{-1}(s) = |\alpha_\kappa^{-1}(E)|$ for any Borel subset E of $[0, 1]$. For any $0 < s \leqslant 1$,

$$\mu_\kappa([0, s]) = \int_0^s t\, d\alpha_\kappa^{-1}(t) = \int_0^\infty \chi_{[0,s]}(\alpha_\kappa(z))\alpha_\kappa(z)\, dz;$$

hence,

$$\mu_\kappa([0, 1]) = \int_0^\infty \alpha_\kappa(z)\, dz = \int_0^\infty |\{s;\ \kappa'(s) \geqslant\}|\, dz = \int_0^\infty \kappa'(s)\, ds = \kappa(1) = 1.$$

By the dominated convergence theorem $\mu_\kappa(\{0\}) = \lim_{s\to 0} \mu_\kappa([0, s]) = 0$. We show that for any function $f \in RL^\infty(T)$, $\|f\|_\kappa = \|f\|_{\mu_\kappa}$,

$$\|f\|_\kappa = \int_{T\times R} \kappa'\big(2d(\Gamma_y(f), x)\big)\, dx\, dy = \int_{T\times R_+} \big|\big\{y;\ \kappa'\big(2d\big(\Gamma_y(f), x\big)\big) \geqslant z\big\}\big|\, dx\, dz$$

$$= \int_{T\times R_+} \big|\big\{y;\ d\big(\Gamma_y(f), x\big) \leqslant \alpha_\kappa(z)/2\big\}\big|\, dx\, dz$$

because of the property (**). Notice that $|\{y;\ d(\Gamma_y(f), x) \leqslant s\}| = \Omega_f(x, s)$. We then get

$$\|f\|_\kappa = \int_{T\times R_+} \Omega_f(x, \alpha_\kappa(z)/2)\, dz\, dx = \int_{T\times[0,1]} \frac{1}{s}\Omega_f(x, s/2)\, d\mu_\kappa(s)\, dx,$$

since $\mu_\kappa(\{0\}) = 0$. We now prove the second part of the theorem. Let $\kappa \in K_0$ and $\mu = \mu_\kappa$. We know that μ is a probability measure and $\mu(\{0\}) = 0$. We set

$$\kappa_\mu(s) = \int_0^s \int_t^1 \frac{1}{\tau} d\mu(\tau)\, dt.$$

By property (***) and the definition of $\mu = \mu_\kappa$,

$$\kappa_\mu(s) = \int_0^s \int_t^1 d\alpha_\kappa^{-1}(\tau)\, d\tau\, dt = \int_0^s |\alpha_\kappa^{-1}([t, 1])|\, dt = \int_0^s \kappa'(t)\, dt = \kappa(s).$$

This shows that $\mu = \mu_\kappa \in P_0$ and $\kappa_{\mu_\kappa} = \kappa_\mu = \kappa$.

Assume now that $\mu \in P_0$ and let

$$\kappa(s) = \kappa_\mu(s) = \int_0^s \int_t^1 \frac{1}{\tau} d\mu(\tau)\, dt.$$

One can see that $\kappa \in K_0$. We next have $\kappa'(s) = \int_s^1 d\mu(\tau)/\tau$, and by property (***), $\kappa'(s) = \int_s^1 d\alpha_\kappa^{-1}(\tau) = \int_s^1 d\mu_\kappa(\tau)/\tau$. Consequently, $\mu = \mu_\kappa$ since also $\mu(\{0\}) = \mu_\kappa(\{0\}) = 0$. This shows that the map $\mu \to \kappa_\mu$ is the inverse of the map $\kappa \to \mu_\kappa$.

REMARKS. If $\kappa \in K_0$ and κ' is continuous strictly decreasing and $\kappa'(1) = 0$, then α_κ is the inverse function of κ'. If $\kappa \in K_0$ is twice differentiable and κ' is strictly decreasing then $d\mu_\kappa(s) = \kappa'(1)\, d\delta_1(s) - s\kappa''(s)\, ds$ where

$$\delta_1(E) = \begin{cases} 1 & \text{if } 1 \in E, \\ 0 & \text{if } 1 \in E. \end{cases}$$

For example, if

(1) $\kappa(s) = s^\alpha$, $0 < \alpha < 1$ ($\kappa(E)$ is the Lipschitz entropy), then $d\mu_\kappa(s) = \alpha\, d\delta_1(s) + \alpha(1-\alpha)s^{\alpha-1}\, ds$;

(2) $\kappa(s) = s(|\log s| + 1)$ ($\kappa(E)$ is the Shannon entropy), then $d\mu_\kappa(s) = ds$;

(3) $\kappa(s) = (1 + \frac{1}{2}|\log s|)^{-1}$ ($\kappa(E)$ is the Dini entropy), then

$$d\mu_\kappa(s) = \frac{1}{2} d\delta_1(s) + \frac{1}{2} \frac{|\log s|}{s(1 + |\log s|/2)^3} ds.$$

COROLLARY. $\|\cdot\|_\kappa$ *is homogeneous and satisfies the triangle inequality for any* $\kappa \in K_0$. *Moreover, if* f *is a real continuously differentiable function on* T, *then* $\|f\|_\kappa \leqslant \|f'\|_\infty \kappa(\|f\|_C/\|f'\|_\infty)$.

PROOF. The first part of the Corollary follows directly from the probability representation of the κ-norms and from the lemma. For the proof of the second part we let f be a nonconstant differentiable function on T, so that $\|f'\|_\infty < \infty$. We denote $A = \|f\|_C$ and $B = \|f'\|_\infty$ ($B \neq 0$). Using the probability representation of the κ-norm we obtain

$$\|f\|_\kappa = \int_{T\times[0,1]} \frac{\Omega_f(t, s/2)}{s} d\mu(s)\, dt = B\int_{T\times[0,1]} \frac{\Omega_{f/B}(t, s/2)}{s} d\mu(s)\, dt$$

$$= B\int_T \int_0^\infty \Omega_{f/B}(t, \alpha_\kappa(y)/2)\, dy\, dt$$

where α_κ is the function defined in the proof of Theorem 1. We notice that $\Omega_{f/B}(t, s/2)$ is nondecreasing and right-continuous in s. We define $\beta_t(y) = \min\{s; \Omega_{f/B}(t, s/2) \geqslant y\}$ for $0 \leqslant y \leqslant A/B$. Clearly $\Omega_{f/B}(t, s/2) \geqslant y$ if and only if $s \geqslant \beta_t(y)$. Moreover, $\beta_t(y) \geqslant y$ for all $0 \leqslant y \leqslant A/B$. Therefore

$$\|f\|_\kappa = B\int_T\int_0^{A/B} |\{z; \alpha_\kappa(z) \geqslant \beta_t(y)\}|\, dy\, dx = B\int_T\int_0^{A/B} \kappa'(\beta_t(y))\, dy\, dt$$
$$\leqslant B\int_T\int_0^{A/B} \kappa'(y)\, dy\, dt = B\kappa(A/B).$$

In what follows κ is again a function from the set K_0. We define the following linear spaces with the norm $\|\cdot\|_\infty + \|\cdot\|_\kappa$; $RL_\kappa^\infty(T) = \{f \in RL^\infty(T); \|f\|_\kappa < \infty\}$ and $RC_\kappa = \{f;\ f \text{ is continuous on } T, \|f\|_\kappa < \infty\}$. The analogous spaces L_κ^∞ and C_κ of complex valued functions are defined as the complexification of the real spaces RL_κ^∞ and RC_κ, respectively (see, for example, [4]). We will use the probability measure representation of κ-norms to prove some general properties of L_κ^∞ and C_κ.

THEOREM 2. (a) *The spaces* L_κ^∞ *and* C_κ *are Banach algebras with the usual multiplication of functions.*

(b) C_κ *is the largest translation invariant subspace of* L_κ^∞ *on which the shift operator* $T_x f(t) = f(t-x)$ *has the property* $\|T_x f - f\|_\kappa \to 0$, *if* $x \to 0$.

(c) *The polynomials are dense in* C_κ. *In particular*, C_κ *is separable.*

PROOF. (a) It is enough to show that $(RL_\kappa^\infty, \|\cdot\|_\infty + \|\cdot\|_\kappa)$ is a Banach algebra. Submultiplicativity of the norm $\|\cdot\|_\infty + \|\cdot\|_\kappa$ follows directly form Theorem 1 and (iii) of the Lemma. To show completeness of $RL_\kappa^\infty(T)$ take a sequence $\{f_n\}_{n=1}^\infty$ in RL_κ^∞ which is Cauchy in the norm $\|\cdot\|_\infty + \|\cdot\|_\kappa$. If f is the uniform limit of f_n then $\Omega_{f_m - f_n}(t, s) \to \Omega_{f-f_n}(t, s)$ as $m \to \infty$ uniformly in (t, s). By Fatou's lemma,

$$\|f - f_n\|_\kappa = \int_{T\times[0,1]} \Omega_{f-f_n}(t, s/2)\frac{d\mu_\kappa(s)}{s}\,dt \leqslant \varliminf_{m\to\infty} \int_{T\times[0,1]} \Omega_{f_m-f_n}(t, s/2)\frac{d\mu_\kappa(s)}{s}\,dt$$
$$= \lim_{m\to\infty} \|f_m - f_n\|_\kappa$$

which shows that $\|f - f_n\|_\kappa \to 0$ as $n \to \infty$.

(b) Let $f \in RC_\kappa$. We must prove that $\lim_{x\to 0}\|T_x f - f\|_\kappa = 0$. Pick $0 < \delta < 1$.

$$\varlimsup_{x\to 0} \|T_x f - f\|_\kappa = \varlimsup_{x\to 0}\left(\int_T\int_0^\delta \Omega_{T_xf-f}(t, s/2)\frac{d\mu_\kappa(s)}{s}\,dt\right)$$
$$+ \varlimsup_{x\to 0}\left(\int_T\int_\delta^1 \Omega_{T_xf-f}(t, s/2)\frac{d\mu_\kappa(s)}{s}\right)dt$$
$$\leqslant \varlimsup_{x\to 0}\int_T\int_0^\delta\left(\Omega_{T_xf} + \Omega_f\right)\frac{d\mu_\kappa(s)}{s}\,dt + \varlimsup_{x\to 0}\left(\int_T\int_\delta^1 \Omega_{T_xf-f}\frac{d\mu_\kappa(s)}{s}\,dt\right)$$
$$= 2\int_T\int_0^\delta \Omega_f\frac{d\mu_\kappa(s)}{s}\,dt.$$

By the dominated convergence theorem,

$$\overline{\lim_{x\to 0}}\, \|T_x f - f\|_\kappa \leqslant 2\, \overline{\lim_{\delta\to 0}} \int_T \int_0^\delta \Omega_f \frac{d\mu_\kappa(s)}{s}\, dt = 0.$$

(c) By part (b) we have that C_κ is a homogeneous Banach space on T [**5**, Definition 2.10] and, consequently, polynomials are dense in it [**5**, Theorem 2.12].

REMARK. We will give the following probabilistic interpretation of the μ-norms. We will pick an interval $I = \{\tau;\, d(t, \tau) < s/2\}$ in T; the center t of I is chosen with the probability evenly distributed along T and the length $|I| = s$ is chosen with the probability of the distibution $\mu(s) = \int_0^s d\mu(\tau)$. The μ-norm of a function f on T is simply the expectation of the random variable $X(I)$ = mean oscillation of f on $I = \Omega_f(t, s/2)/s$ in this process.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, ALBANY, NEW YORK 12222

DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY, WROCŁAW, POLAND