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REFLEXIVITY OF A BANACH SPACE WITH A UNIFORMLY NORMAL STRUCTURE

JONG SOOK BAE

ABSTRACT. In this note we prove that any Banach space with a uniformly normal structure is reflexive.

1. Introduction. In [4] Gillespie and Williams gave the concept of uniformly normal structure of Banach spaces. They showed that any nonexpansive self-map of a closed convex bounded subset of a Banach space with a uniformly normal structure has a fixed point, and, in [5], obtained the same result for the Kannan-type maps.

In this note, we show that any Banach space with a uniformly normal structure is reflexive, and, consequently, the main results of Gillespie and Williams are actually contained in those of Kirk [9], Godhe [6] and Kannan [8], respectively.

2. Main result. A Banach space X is said to have a uniformly normal structure if there exists a number h, $0 < h < 1$, such that if C is a closed convex bounded subset of X, then there exists x in C such that $\sup\{\|x - y\|; y \in C\} \leqslant h\delta(C)$, where $\delta(C)$ denotes the diameter of the set C.

To prove our theorem, we adopt the idea of Huff [7].

THEOREM. *Any Banach space with a uniformly normal structure is reflexive.*

PROOF. We use a theorem of Eberlein and Smulian [2, p. 51]. Let $\{K_n\}$ be a decreasing sequence of nonvoid closed convex bounded subsets of a given Banach space with a uniformly normal structure. We need to show that $\bigcap K_n \neq \varnothing$. For each n, choose $x_n \in K_n$. Call a sequence $\{y_n\}$ a c-subsequence of $\{x_n\}$ provided there exists a sequence of integers $1 = p_1 \leqslant q_1 < p_2 \leqslant q_2 < \cdots$ and coefficients $\alpha_i \geqslant 0$ such that, for each n,

$$\sum_{i=p_n}^{q_n} \alpha_i = 1, \qquad y_n = \sum_{i=p_n}^{q_n} \alpha_i x_i.$$

Then for each $\varepsilon > 0$, there exists a c-subsequence $\{y_n\}$ of $\{x_n\}$ with $\|y_n - y_m\| < \varepsilon$ for each n, m. Suppose this is not true for some $\varepsilon > 0$.

Let $L_m = \{x_n\}_{n=m}^{\infty}$. Let $\operatorname{co}(L_m)$ and $\overline{\operatorname{co}}(L_m)$ denote the convex hull and the closed convex hull of L_m, respectively. Then there exists h, $0 < h < 1$, and $y_1' \in \overline{\operatorname{co}}(L_1)$ such

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that $\sup\{\|y_1' - y\|;\ y \in \overline{\mathrm{co}}(L_1)\} \leqslant h\delta(L_1)$. Let $0 < h < h_1 < 1$. Then by the triangle inequality there exists $y_1 \in \mathrm{co}(L_1)$ such that $\sup\{\|y_1 - y\|;\ y \in \overline{\mathrm{co}}(L_1)\} \leqslant h_1\delta(L_1)$. Since y_1 is a finite linear combination of members in L_1, there exists a c-subsequence $\{y_n\}$ of $\{x_n\}$ such that $\sup\{\|y_n - y\|;\ y \in \overline{\mathrm{co}}(L_{p_n})\} \leqslant h_1\delta(L_{p_n}) \leqslant h_1\delta(L_1)$, and this inequality shows that $\delta(\{y_n\}) \leqslant h_1\delta(L_1)$. By repeating the argument, there exists a successive c-subsequence with diameter less than or equal to $h_1^2\delta(L_1)$. We need only repeat the argument a sufficient number k of times with $h_1^k\delta(L_1) < \varepsilon$ to obtain a contradiction.

Next by the diagonal method, there exists a c-subsequence of $\{x_n\}$ which is norm Cauchy, and hence convergent to some y. Then $y \in \bigcap K_n$.

REMARKS. Bynum [**1**] showed that a uniformly convex Banach space has a uniformly normal structure. But the converse is not true. For example, the space l_2, renormed by

$$\|(x_i)\|_1 = \max\left\{|x_1|,\ \left(\sum_{i=2}^{\infty}|x_i|^2\right)^{1/2}\right\}$$

has a uniformly normal structure, but $(l_2, \|\ \|_1)$ is not uniformly convex.

Bynum [**1**] also showed that there exists a reflexive space with normal structure, but without a uniformly normal structure.

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DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, DAEJEON, SEOUL, KOREA