

A NOTE ON AN OSCILLATION CRITERION FOR AN EQUATION WITH DAMPED TERM

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ABSTRACT. A new oscillation criterion is given for the equation $x''(t) + p(t)x'(t) + q(t)x(t) = 0$, $t \in [t_0, \infty)$, where $p(t)$ and $q(t)$ are allowed to change sign on $[t_0, \infty)$.

Let us consider the second order differential equation with damped term

$$(1) \quad x''(t) + p(t)x'(t) + q(t)x(t) = 0,$$

and the more general equation

$$(1)' \quad x''(t) + p(t)x'(t) + q(t)f(x(t)) = 0,$$

where $p, q \in C[t_0, \infty)$ and are allowed to assume negative values for arbitrarily large t , $f \in C(\mathbb{R})$, $xf(x) > 0$ for $x \neq 0$.

We shall restrict our attention to solutions of (1) or (1)' which exist on some ray $[\tilde{t}, \infty)$. A solution of an equation is called oscillatory if it has no largest zero; otherwise it is nonoscillatory. An equation is said to be oscillatory if every solution is oscillatory.

For the second order linear differential equation

$$(*) \quad x''(t) + q(t)x(t) = 0,$$

Wintner [6] proved that a sufficient condition for oscillation was

$$(**) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \int_{t_0}^s q(\tau) d\tau ds = \infty.$$

Hartman [3] proved that the limit cannot be replaced by the upper limit in condition (**) and

$$-\infty < \liminf_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \int_{t_0}^s q(\tau) d\tau ds < \limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \int_{t_0}^s q(\tau) d\tau ds \leq \infty$$

implies (*) is oscillatory.

Later, important developments by Willett and Coles in averaging techniques for oscillation of (*) were made. Willett [5] and Coles [2], respectively, established more general theorems by considering weighted averages of the integral of q .

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Several years ago Kamenev [4] obtained an oscillation criterion for (*), namely, (*) is oscillatory if for some $n > 2$,

$$\limsup_{t \rightarrow \infty} t^{1-n} \int_{t_0}^t (t-s)^{n-1} q(s) ds = \infty,$$

which extended Wintner's result.

Recently, Yeh [8] has shown some oscillation criteria of (1)' by using a technique similar to Kamenev's, which included results of [1, 4 and 6].

The purpose of this note is to proceed further in this direction and present a new oscillation theorem which improves Kamenev's criterion. A more general version of the theorem contains the theorems of Yeh [7 and 8].

Our result is as follows:

THEOREM. *Suppose for some $\alpha \in (1, \infty)$ and $\beta \in [0, 1)$,*

$$(2) \quad \limsup_{t \rightarrow \infty} t^{-\alpha} \int_{t_0}^t (t-s)^\alpha s^\beta q(s) ds = \infty,$$

$$(3) \quad \limsup_{t \rightarrow \infty} t^{-\alpha} \int_{t_0}^t [(t-s)p(s)s + \alpha s - \beta(t-s)]^2 (t-s)^{\alpha-2} s^{\beta-2} ds < \infty.$$

Then (1) is oscillatory.

PROOF. Assume the contrary. Then (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume $x(t) \neq 0$ for $t \geq t_0$. Define $\omega(t) = x'(t)/x(t)$. Then it follows from (1) that

$$\omega'(t) + \omega^2(t) + p(t)\omega(t) + q(t) = 0.$$

Hence

$$\begin{aligned} \int_{t_0}^t (t-s)^\alpha s^\beta \omega'(s) ds + \int_{t_0}^t (t-s)^\alpha s^\beta \omega^2(s) ds \\ + \int_{t_0}^t (t-s)^\alpha s^\beta p(s)\omega(s) ds + \int_{t_0}^t (t-s)^\alpha s^\beta q(s) ds \leq 0. \end{aligned}$$

Noting that

$$\begin{aligned} \int_{t_0}^t (t-s)^\alpha s^\beta \omega'(s) ds &= \alpha \int_{t_0}^t (t-s)^{\alpha-1} s^\beta \omega(s) ds - \beta \int_{t_0}^t (t-s)^\alpha s^{\beta-1} \omega(s) ds \\ &\quad - \omega(t_0)(t-t_0)^\alpha t_0^\beta, \end{aligned}$$

we obtain

$$\begin{aligned} \int_{t_0}^t (t-s)^\alpha s^\beta q(s) ds \leq \omega(t_0)(t-t_0)^\alpha t_0^\beta - \int_{t_0}^t (t-s)^\alpha s^\beta \omega^2(s) ds \\ - \int_{t_0}^t [(t-s)p(s)s + \alpha s - \beta(t-s)] (t-s)^{\alpha-1} s^{\beta-1} \omega(s) ds. \end{aligned}$$

Dividing by t^α and taking the upper limit as $t \rightarrow \infty$, we get

$$\begin{aligned} & \limsup_{t \rightarrow \infty} t^{-\alpha} \int_{t_0}^t (t-s)^\alpha s^\beta q(s) ds \\ & \leq \omega(t_0)t_0^\beta + \limsup_{t \rightarrow \infty} \frac{t^{-\alpha}}{4} \int_{t_0}^t [(t-s)sp(s) + \alpha s - \beta(t-s)]^2 (t-s)^{\alpha-2} s^{\beta-2} ds \\ & \quad - \liminf_{t \rightarrow \infty} t^{-\alpha} \int_{t_0}^t \{ (t-s)^{\alpha/2} s^{\beta/2} \omega(s) \\ & \quad \quad + \frac{1}{2} [(t-s)sp(s) + \alpha s - \beta(t-s)](t-s)^{(\alpha-2)/2} s^{(\beta-2)/2} \}^2 ds < \infty, \end{aligned}$$

which contradicts conditions (2) and (3). This completes the proof.

Let $p(t) = 0$. Then (3) is satisfied automatically. Thus we have

COROLLARY 1. *Suppose for some $\alpha \in (1, \infty)$ and $\beta \in [0, 1)$, (2) is satisfied. Then (1) is oscillatory.*

REMARK 1. Corollary 1 improves and generalizes Kamenev's theorem [4].

From the proof of the theorem, we easily obtain the following extension to (1)'.

COROLLARY 2. *Suppose*

$$(4) \quad f'(x) \text{ exists and } f'(x) \geq k > 0$$

for some constant k and for all $x \neq 0$. If (2) and (3) hold, then (1)' is oscillatory.

Taking $\alpha = n - 1, \beta = 0$ in (2) and (3), we get

COROLLARY 3. *Suppose (4) is satisfied. If*

$$(2)' \quad \limsup_{t \rightarrow \infty} t^{1-n} \int_{t_0}^t (t-s)^{n-1} q(s) ds = \infty$$

and

$$(3)' \quad \limsup_{t \rightarrow \infty} t^{1-n} \int_{t_0}^t [(t-s)p(s) + (n-1)]^2 (t-s)^{n-3} ds < \infty$$

for some $n > 2$ (not necessarily integral), then (1)' is oscillatory.

REMARK 2. Corollary 3 includes Kamenev's [4] and Yeh's theorem [7 and 8].

As an example, the equation

$$(5) \quad x''(t) + \frac{\sin t}{t^\mu} x'(t) + \frac{\cos t}{t^\nu} x(t) = 0, \quad 1 \leq \mu < \infty, 0 \leq \nu < 1.$$

Taking $\alpha = 2, \nu < \beta < 1$, we easily verify that all conditions of our theorem are satisfied. Hence, (5) is oscillatory. However, each of the criteria in [4, 7 and 8] fail to apply to (5). On the other hand, (5) cannot be reduced to a form in which some other known results may be used.

We could establish corresponding theorems by the method that is used in this note, which would improve other results of [8].

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