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A NOTE ON AN OSCILLATION CRITERION FOR AN EQUATION WITH DAMPED TERM

JURANG YAN

ABSTRACT. A new oscillation criterion is given for the equation $x''(t) + p(t)x'(t) + q(t)x(t) = 0$, $t \in [t_0, \infty)$, where $p(t)$ and $q(t)$ are allowed to change sign on $[t_0, \infty)$.

Let us consider the second order differential equation with damped term

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0, \tag{1}$$

and the more general equation

$$x''(t) + p(t)x'(t) + q(t)f(x(t)) = 0, \tag{1$'$}$$

where $p, q \in C[t_0, \infty)$ and are allowed to assume negative values for arbitrarily large t, $f \in C(R)$, $xf(x) > 0$ for $x \neq 0$.

We shall restrict our attention to solutions of (1) or (1)$'$ which exist on some ray $[\tilde{t}, \infty)$. A solution of an equation is called oscillatory if it has no largest zero; otherwise it is nonoscillatory. An equation is said to be oscillatory if every solution is oscillatory.

For the second order linear differential equation

$$x''(t) + q(t)x(t) = 0, \tag{$*$}$$

Wintner [6] proved that a sufficient condition for oscillation was

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)\,d\tau\,ds = \infty. \tag{$**$}$$

Hartman [3] proved that the limit cannot be replaced by the upper limit in condition $(**)$ and

$$-\infty < \lim_{t\to\infty} \inf \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)\,d\tau\,ds < \lim_{t\to\infty} \sup \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)\,d\tau\,ds \leqslant \infty$$

implies $(*)$ is oscillatory.

Later, important developments by Willett and Coles in averaging techniques for oscillation of $(*)$ were made. Willett [5] and Coles [2], respectively, established more general theorems by considering weighted averages of the integral of q.

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Several years ago Kamenev [4] obtained an oscillation criterion for (*), namely, (*) is oscillatory if for some $n > 2$,

$$\lim_{t\to\infty} \sup t^{1-n}\int_{t_0}^{t}(t-s)^{n-1}q(s)\,ds = \infty,$$

which extended Wintner's result.

Recently, Yeh [8] has shown some oscillation criteria of (1)′ by using a technique similar to Kamenev's, which included results of [1, 4 and 6].

The purpose of this note is to proceed further in this direction and present a new oscillation theorem which improves Kamenev's criterion. A more general version of the theorem contains the theorems of Yeh [7 and 8].

Our result is as follows:

THEOREM. *Suppose for some* $\alpha \in (1,\infty)$ *and* $\beta \in [0,1)$,

$$\lim_{t\to\infty} \sup t^{-\alpha}\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}q(s)\,ds = \infty, \tag{2}$$

$$\lim_{t\to\infty} \sup t^{-\alpha}\int_{t_0}^{t}[(t-s)p(s)s+\alpha s-\beta(t-s)]^2(t-s)^{\alpha-2}s^{\beta-2}\,ds < \infty. \tag{3}$$

Then (1) *is oscillatory.*

PROOF. Assume the contrary. Then (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume $x(t)\neq 0$ for $t \geqslant t_0$. Define $\omega(t) = x'(t)/x(t)$. Then it follows from (1) that

$$\omega'(t)+\omega^2(t)+p(t)\omega(t)+q(t)=0.$$

Hence

$$\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}\omega'(s)\,ds+\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}\omega^2(s)\,ds \\ +\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}p(s)\omega(s)\,ds+\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}q(s)\,ds \leqslant 0.$$

Noting that

$$\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}\omega'(s)\,ds = \alpha\int_{t_0}^{t}(t-s)^{\alpha-1}s^{\beta}\omega(s)\,ds-\beta\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta-1}\omega(s)\,ds \\ -\omega(t_0)(t-t_0)^{\alpha}t_0^{\beta},$$

we obtain

$$\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}q(s)\,ds \leqslant \omega(t_0)(t-t_0)^{\alpha}t_0^{\beta}-\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}\omega^2(s)\,ds \\ -\int_{t_0}^{t}[(t-s)p(s)s+\alpha s-\beta(t-s)](t-s)^{\alpha-1}s^{\beta-1}\omega(s)\,ds.$$

Dividing by t^α and taking the upper limit as $t \to \infty$, we get

$$\begin{aligned}
&\limsup_{t\to\infty} t^{-\alpha}\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}q(s)\,ds \\
&\quad\leq \omega(t_0)t_0^{\beta} + \limsup_{t\to\infty}\frac{t^{-\alpha}}{4}\int_{t_0}^{t}[(t-s)sp(s)+\alpha s-\beta(t-s)]^2(t-s)^{\alpha-2}s^{\beta-2}\,ds \\
&\qquad - \liminf_{t\to\infty} t^{-\alpha}\int_{t_0}^{t}\{(t-s)^{\alpha/2}s^{\beta/2}\omega(s) \\
&\qquad\qquad + \tfrac{1}{2}[(t-s)sp(s)+\alpha s-\beta(t-s)](t-s)^{(\alpha-2)/2}s^{(\beta-2)/2}\}^2\,ds < \infty,
\end{aligned}$$

which contradicts conditions (2) and (3). This completes the proof.

Let $p(t) = 0$. Then (3) is satisfied automatically. Thus we have

COROLLARY 1. *Suppose for some $\alpha \in (1,\infty)$ and $\beta \in [0,1)$,* (2) *is satisfied. Then* (1) *is oscillatory.*

REMARK 1. Corollary 1 improves and generalizes Kamenev's theorem [**4**].

From the proof of the theorem, we easily obtain the following extension to (1)′.

COROLLARY 2. *Suppose*

(4) $$f'(x) \text{ exists} \quad \text{and} \quad f'(x) \geq k > 0$$

for some constant k and for all $x \neq 0$. If (2) *and* (3) *hold, then* (1)′ *is oscillatory.*

Taking $\alpha = n - 1$, $\beta = 0$ in (2) and (3), we get

COROLLARY 3. *Suppose* (4) *is satisfied. If*

(2)′ $$\limsup_{t\to\infty} t^{1-n}\int_{t_0}^{t}(t-s)^{n-1}q(s)\,ds = \infty$$

and

(3)′ $$\limsup_{t\to\infty} t^{1-n}\int_{t_0}^{t}[(t-s)p(s)+(n-1)]^2(t-s)^{n-3}\,ds < \infty$$

for some $n > 2$ (not necessarily integral), then (1)′ *is oscillatory.*

REMARK 2. Corollary 3 includes Kamenev's [**4**] and Yeh's theorem [**7** and **8**].

As an example, the equation

(5) $$x''(t) + \frac{\sin t}{t^{\mu}}x'(t) + \frac{\cos t}{t^{\nu}}x(t) = 0, \qquad 1 \leq \mu < \infty, 0 \leq \nu < 1.$$

Taking $\alpha = 2$, $\nu < \beta < 1$, we easily verify that all conditions of our theorem are satisfied. Hence, (5) is oscillatory. However, each of the criteria in [**4**, **7** and **8**] fail to apply to (5). On the other hand, (5) cannot be reduced to a form in which some other known results may be used.

We could establish corresponding theorems by the method that is used in this note, which would improve other results of [**8**].

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Department of Mathematics, Shanxi University, Taiyuan, Shanxi, People's Republic of China (Current address)

Department of Mathematics and Statistics, University of South Carolina, Columbia, South Carolina 29208