

CONTINUITY OF MEASURABLE CONVEX AND BICONVEX OPERATORS

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ABSTRACT. We prove that a mapping from the product of two complete metrizable vector spaces into a topological vector space which is separately universally measurable and separately convex with respect to a convex cone is continuous.

0. Introduction. If X is a complete separable vector space and if f is a midpoint convex mapping from X into the *real line* \mathbf{R} which is Christensen measurable, then F. Fischer and Z. Slodkowski have proved in [4] that f is continuous by showing that $\{(x, r) \in X \times \mathbf{R} : f(x) < r\}$ is open in $X \times \mathbf{R}$. Results of this kind about continuity of universally measurable morphisms were obtained before by A. Douady and L. Schwartz (see [7]) for linear operators between locally convex spaces, and by J. P. R. Christensen in [3] for homomorphisms between topological complete metrizable groups.

In this paper we are concerned with the study of convex operators taking values in a vector space. Vector analogues of the preceding results are proved. In fact, we establish that a separately universally measurable mapping $f: X \times Y \rightarrow F$, where X and Y are two complete metrizable vector spaces and F is any topological vector space, is continuous whenever it is separately midpoint convex with respect to a convex cone.

Let us also note that many other interesting results about continuity of convex operators are given by J. M. Borwein in [1].

1. Preliminaries. Throughout this paper F will denote a (real separated) topological vector space and X and Y two real *complete metrizable* topological vector spaces.

Let P be a *convex cone* in F , i.e. $tP + sP \subset P$ for all $t, s \geq 0$. One says that P is *normal* if there is a base of neighbourhoods V of zero with

$$V = (V + P) \cap (V - P).$$

Such neighbourhoods are said to be *full*. Many properties and examples of normal convex cones can be found in [7]. In the sequel we shall always assume that P is a *normal convex cone* in F .

A subset B of a topological space S is *universally measurable* if for each finite measure m over the Borel tribe $\mathfrak{B}(S)$ the set B belongs to the m -completion of $\mathfrak{B}(S)$. The set of universally measurable subsets of S will be denoted by $\mathfrak{U}(S)$.

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A mapping $f: S \rightarrow F$ is universally measurable if for each open subset Ω of F the set $f^{-1}(\Omega)$ is in $\mathcal{Q}(S)$.

1.1. **REMARK.** As a direct consequence of Proposition 8 in [5, p. 12], for each continuous mapping f from a topological space S into a topological space T , one has $f^{-1}(B) \in \mathcal{Q}(S)$ for each $B \in \mathcal{Q}(T)$. \square

Let us recall the following consequence of a very nice and important result of Christensen.

1.2. **PROPOSITION.** *If $X = \bigcup_{n \in \mathbb{N}} B_n$ is a countable union of universally measurable subsets, then there exists an integer k such that $B_k - B_k = \{x - y: x, y \in B_k\}$ is a neighbourhood of zero.*

PROOF. This is a direct consequence of Theorem 7.1 in [3]. \square

2. Midpoint convex operators.

2.1. **DEFINITION.** One says that $f: X \rightarrow F$ is *P-convex* if

$$(2.1) \quad f(tx + (1-t)y) \in tf(x) + (1-t)f(y) - P \quad \text{for all } t \in [0, 1],$$

and f is *midpoint P-convex* if (2.1) holds for $t = 1/2$.

One observes that additive mappings are always midpoint convex.

REMARK. It is easily shown that f is midpoint *P-convex* if and only if

$$(2.2) \quad f(2^{-n}kx + (1-2^{-n}k)y) \in 2^{-n}kf(x) + (1-2^{-n}k)f(y) - P$$

for all $x, y \in X$, k and $n \in \mathbb{N}$ with $0 \leq k \leq 2^n$. It follows that a midpoint *P-convex* operator with closed epigraph is convex.

2.2. **LEMMA.** *A midpoint P-convex operator $f: X \rightarrow F$ is continuous at $a \in X$ if and only if f is upper semicontinuous at a in the following sense: for every neighbourhood W of zero in F there exists a neighbourhood V of zero in X such that*

$$f(a+x) \in f(a) + W - P \quad \text{for each } x \in V.$$

PROOF. It is clearly enough to show that the condition is sufficient. Let W be any neighbourhood of zero in F . Choose a full circled neighbourhood W_0 of zero in Y with $W_0 \subset W$ and a circled neighbourhood V of zero in X satisfying

$$(2.3) \quad f(a+x) - f(a) \in W_0 - P \quad \text{for all } x \in V.$$

As f is midpoint *P-convex*, we have for each $x \in V$

$$f(a) \in \frac{1}{2}f(a-x) + \frac{1}{2}f(a+x) - P$$

and hence

$$f(a+x) - f(a) \in f(a) - f(a-x) + P \subset W_0 + P + P = W_0 + P.$$

Making use of relation (2.3) once again we obtain

$$f(a+x) - f(a) \in (W_0 - P) \cap (W_0 + P) = W_0 \subset W$$

for each $x \in V$. \square

2.3. **PROPOSITION.** *Let $f: X \rightarrow F$ be a universally measurable midpoint P-convex operator; then:*

- (i) f is continuous,
- (ii) if P is closed, f is a continuous *P-convex* operator.

PROOF. Let us begin by proving the continuity. Let a be any point in X . Put $g(x) = f(a + x) - f(a)$. The operator g is obviously a midpoint convex operator with $g(0) = 0$ and by Remark 1.1 it is universally measurable. Let W_0 be a neighbourhood of zero in F . Choose an open circled neighbourhood W of zero with $W + W \subset W_0$. For each nonnegative integer n consider the universally measurable set

$$B_n = \{x \in X: g(x) \in 2^{n+1}W, g(-x) \in 2^{n+1}W\}.$$

Since $X = \bigcup_{n \in \mathbb{N}} B_n$, there exists, by Proposition 1.2, an integer k such that $B_k - B_k$ is a neighbourhood of zero in X , and for each x in the neighbourhood of zero $V := \frac{1}{2}B_k - \frac{1}{2}B_k = \frac{1}{2}B_k + \frac{1}{2}B_k$ we may write $x = \frac{1}{2}b + \frac{1}{2}b'$ with $b, b' \in B_k$,

$$g(x) \in \frac{1}{2}g(b) + \frac{1}{2}g(b') - P \subset 2^k W + 2^k W - P \subset 2^k W_0 - P$$

and hence

$$2^{-k}g(x) \in W_0 - P.$$

As g is midpoint P -convex with $g(0) = 0$, we have for each $x \in V$,

$$g(2^{-k}x) \in 2^{-k}g(x) - P \subset W_0 - P$$

and hence by Lemma 2.2, g is continuous at 0, which implies that f is continuous.

If P is closed, then the continuity of f and relation (2.2) easily imply that f is P -convex. \square

3. Midpoint biconvex operators.

3.1. DEFINITION. A mapping $f: X \times Y \rightarrow F$ is called a *midpoint P -biconvex operator* if for each $(x, y) \in X \times Y$ the mappings $f(x, \cdot)$ and $f(\cdot, y)$ are midpoint P -convex operators.

3.2. PROPOSITION. *Let $f: X \times Y \rightarrow F$ be a separately universally measurable midpoint P -biconvex operator; then:*

- (i) f is continuous,
- (ii) if P is closed, f is a continuous P -biconvex operator.

PROOF. Let (c, d) be any point in $X \times Y$. Put $g(x, y) = f(c + x, d + y) - f(c, d + y)$. The mapping g is separately universally measurable and $g(0, 0) = 0$. Let $(x_n, y_n)_{n \in \mathbb{N}}$ be any sequence in $X \times Y$ converging to zero and let W_0 be any full circled neighbourhood of zero in F . Choose an open neighbourhood W of zero satisfying $W + W \subset W_0$. For each $x \in X$, by Proposition 2.3, the mapping $g(x, \cdot)$ is continuous and hence the set $\{g(x, y_n): n \in \mathbb{N}\}$ is topologically bounded in F as $\{0\} \cup \{y_n: n \in \mathbb{N}\}$ is compact in Y . So if we put, for each $p \in \mathbb{N}$,

$$B_p = \{x \in X: g(x, y_n) \in 2^{p+1}W \text{ and } g(-x, y_n) \in 2^{p+1}W, \forall n \in \mathbb{N}\},$$

then $X = \bigcup_{p \in \mathbb{N}} B_p$ and hence, by Proposition 1.2, there exists an integer k and a circled neighbourhood V of zero in X with $V \subset \frac{1}{2}B_k - \frac{1}{2}B_k = \frac{1}{2}B_k + \frac{1}{2}B_k$. Therefore, for each $x = \frac{1}{2}b + \frac{1}{2}b' \in V$ with $b, b' \in B_k$ and each $n \in \mathbb{N}$, invoking the midpoint convexity of $g(\cdot, y_n)$ we have

$$g(x, y_n) \in \frac{1}{2}g(b, y_n) + \frac{1}{2}g(b', y_n) - P \subset 2^k W + 2^k W - P \subset 2^k W_0 - P$$

and hence again, by the midpoint convexity of $g(\cdot, y_n)$ and the relation $g(0, y_n) = 0$, we have

$$g(2^{-k}V, y_n) \subset (W_0 - P) \cap (W_0 + P) = W_0 \quad \text{for every } n \in \mathbb{N}.$$

As $\lim_{n \rightarrow \infty} x_n = 0$, we may conclude that $\lim_{n \rightarrow \infty} g(x_n, y_n) = 0$ and the proof is complete since $y \rightarrow f(c, d + y)$ is continuous. \square

The above proof also gives the following result.

3.3. PROPOSITION. *Let $f: X \times Y \rightarrow F$ be a universally measurable midpoint P -convex-concave operator, that is $f(\cdot, y)$ and $-f(x, \cdot)$ are midpoint P -convex for each $(x, y) \in X \times Y$; then:*

- (i) *f is continuous,*
- (ii) *if P is closed, f is a continuous P -convex-concave operator.*

REMARKS. (1) If X and Y are also *separable* and if $\mathcal{C}(X)$ denotes the tribe of Christensen measurable subsets of X , that is (see [4]) the set of subsets $C \subset X$ for which there exist two universally measurable subsets A and M , a probability measure m on $\mathcal{U}(X)$ and a subset $N \subset M$ such that $C = A \cup N$ and $m(x + M) = 0$ for all $x \in X$, the above result still holds whenever f is separately $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ measurable.

(2) Results in the line of Proposition 3.2 about equicontinuous families of biconvex or concave-convex operators can be found in [6].

REFERENCES

1. J. M. Borwein, *Continuity and differentiability properties of convex operators*, Proc. London Math. Soc. (3) **44** (1982), 420–444.
2. J. P. R. Christensen, *Borel structures in groups and semigroups*, Mat. Scand. **28** (1971), 124–128.
3. ———, *Topology and Borel structure*, North-Holland, Amsterdam and American Elsevier, New York, 1974.
4. P. Fischer and Z. Slodkowski, *Christensen zero sets and measurable convex functions*, Proc. Amer. Math. Soc. **79** (1980), 449–453.
5. J. Hoffmann-Jorgensen, *The theory of analytic sets*, Aarhus Universitet Danmark, various publication series, no. 10, 1970.
6. M. Jouak and L. Thibault, *Equicontinuity of families of convex and concave-convex operators* (to appear).
7. A. L. Peressini, *Ordered topological vector spaces*, Harper & Row, New York, 1971.
8. A. L. Schwartz, *Sur le théorème du graphe fermé*, C. R. Acad. Sci. Paris Sér A **263** (1966), 602–605.
9. L. Thibault, *Continuity of measurable convex multifunctions* (to appear).

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