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CONTINUITY OF MEASURABLE CONVEX AND BICONVEX OPERATORS

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ABSTRACT. We prove that a mapping from the product of two complete metrizable vector spaces into a topological vector space which is separately universally measurable and separately convex with respect to a convex cone is continuous.

0. Introduction. If X is a complete separable vector space and if f is a midpoint convex mapping from X into the *real line* $\mathbf{R}$ which is Christensen measurable, then F. Fischer and Z. Slodkowski have proved in [**4**] that f is continuous by showing that $\{(x, r) \in X \times \mathbf{R}: f(x) < r\}$ is open in $X \times \mathbf{R}$. Results of this kind about continuity of universally measurable morphisms were obtained before by A. Douady and L. Schwartz (see [**7**]) for linear operators between locally convex spaces, and by J. P. R. Christensen in [**3**] for homomorphisms between topological complete metrizable groups.

In this paper we are concerned with the study of convex operators taking values in a vector space. Vector analogues of the preceding results are proved. In fact, we establish that a separately universally measurable mapping f: $X \times Y \to F$, where X and Y are two complete metrizable vector spaces and F is any topological vector space, is continuous whenever it is separately midpoint convex with respect to a convex cone.

Let us also note that many other interesting results about continuity of convex operators are given by J. M. Borwein in [**1**].

1. Preliminaries. Throughout this paper F will denote a (real separated) topological vector space and X and Y two real *complete metrizable* topological vector spaces.

Let P be a *convex cone* in F, i.e. $tP + sP \subset P$ for all $t, s \geqslant 0$. One says that P is *normal* if there is a base of neighbourhoods V of zero with

$$V = (V + P) \cap (V - P).$$

Such neighbourhoods are said to be *full*. Many properties and examples of normal convex cones can be found in [**7**]. In the sequel we shall always assume that P is a *normal convex* cone in F.

A subset B of a topological space S is universally measurable if for each finite measure m over the Borel tribe $\mathcal{B}(S)$ the set B belongs to the m-completion of $\mathcal{B}(S)$. The set of universally measurable subsets of S will be denoted by $\mathcal{U}(S)$.

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A mapping f: $S \to F$ is universally measurable if for each open subset Ω of F the set $f^{-1}(\Omega)$ is in $\mathcal{U}(S)$.

1.1. REMARK. As a direct consequence of Proposition 8 in [5, p. 12], for each continuous mapping f from a topological space S into a topological space T, one has $f^{-1}(B) \in \mathcal{U}(S)$ for each $B \in \mathcal{U}(T)$. □

Let us recall the following consequence of a very nice and important result of Christensen.

1.2. PROPOSITION. *If $X = \bigcup_{n \in \mathbf{N}} B_n$ is a countable union of universally measurable subsets, then there exists an integer k such that $B_k - B_k = \{x - y: x, y \in B_k\}$ is a neighbourhood of zero.*

PROOF. This is a direct consequence of Theorem 7.1 in [3]. □

2. Midpoint convex operators.

2.1. DEFINITION. One says that f: $X \to F$ is *P-convex* if

$$f(tx + (1-t)y) \in tf(x) + (1-t)f(y) - P \quad \text{for all } t \in [0, 1], \tag{2.1}$$

and f is *midpoint P-convex* if (2.1) holds for $t = 1/2$.

One observes that additive mappings are always midpoint convex.

REMARK. It is easily shown that f is midpoint P-convex if and only if

$$f(2^{-n}kx + (1 - 2^{-n}k)y) \in 2^{-n}kf(x) + (1 - 2^{-n}k)f(y) - P \tag{2.2}$$

for all $x, y \in X$, k and $n \in \mathbf{N}$ with $0 \leqslant k \leqslant 2^n$. It follows that a midpoint P convex operator with closed epigraph is convex.

2.2. LEMMA. *A midpoint P-convex operator f: $X \to F$ is continuous at $a \in X$ if and only if f is upper semicontinuous at a in the following sense: for every neighbourhood W of zero in F there exists a neighbourhood V of zero in X such that*

$$f(a + x) \in f(a) + W - P \quad \textit{for each } x \in V.$$

PROOF. It is clearly enough to show that the condition is sufficient. Let W be any neighbourhood of zero in F. Choose a full circled neighbourhood W_0 of zero in Y with $W_0 \subset W$ and a circled neighbourhood V of zero in X satisfying

$$f(a + x) - f(a) \in W_0 - P \quad \text{for all } x \in V. \tag{2.3}$$

As f is midpoint P-convex, we have for each $x \in V$

$$f(a) \in \tfrac{1}{2}f(a - x) + \tfrac{1}{2}f(a + x) - P$$

and hence

$$f(a + x) - f(a) \in f(a) - f(a - x) + P \subset W_0 + P + P = W_0 + P.$$

Making use of relation (2.3) once again we obtain

$$f(a + x) - f(a) \in (W_0 - P) \cap (W_0 + P) = W_0 \subset W$$

for each $x \in V$. □

2.3. PROPOSITION. *Let f: $X \to F$ be a universally measurable midpoint P-convex operator; then:*

(i) *f is continuous,*

(ii) *if P is closed, f is a continuous P-convex operator.*

PROOF. Let us begin by proving the continuity. Let a be any point in X. Put $g(x) = f(a + x) - f(a)$. The operator g is obviously a midpoint convex operator with $g(0) = 0$ and by Remark 1.1 it is universally measurable. Let W_0 be a neighbourhood of zero in F. Choose an open circled neighbourhood W of zero with $W + W \subset W_0$. For each nonnegative integer n consider the universally measurable set

$$B_n = \{x \in X: g(x) \in 2^{n+1}W, g(-x) \in 2^{n+1}W\}.$$

Since $X = \bigcup_{n \in \mathbf{N}} B_n$, there exists, by Proposition 1.2, an integer k such that $B_k - B_k$ is a neighbourhood of zero in X, and for each x in the neighbourhood of zero $V := \frac{1}{2}B_k - \frac{1}{2}B_k = \frac{1}{2}B_k + \frac{1}{2}B_k$ we may write $x = \frac{1}{2}b + \frac{1}{2}b'$ with $b, b' \in B_k$,

$$g(x) \in \tfrac{1}{2}g(b) + \tfrac{1}{2}g(b') - P \subset 2^k W + 2^k W - P \subset 2^k W_0 - P$$

and hence

$$2^{-k}g(x) \in W_0 - P.$$

As g is midpoint P-convex with $g(0) = 0$, we have for each $x \in V$,

$$g(2^{-k}x) \in 2^{-k}g(x) - P \subset W_0 - P$$

and hence by Lemma 2.2, g is continuous at 0, which implies that f is continuous.

If P is closed, then the continuity of f and relation (2.2) easily imply that f is P-convex. □

3. Midpoint biconvex operators.

3.1. DEFINITION. A mapping f: $X \times Y \to F$ is called a *midpoint P-biconvex operator* if for each $(x, y) \in X \times Y$ the mappings $f(x, \cdot)$ and $f(\cdot, y)$ are midpoint P-convex operators.

3.2. PROPOSITION. *Let f: $X \times Y \to F$ be a separately universally measurable midpoint P-biconvex operator; then*:

(i) *f is continuous*,

(ii) *if P is closed, f is a continuous P-biconvex operator*.

PROOF. Let (c, d) be any point in $X \times Y$. Put $g(x, y) = f(c + x, d + y) - f(c, d + y)$. The mapping g is separately universally measurable and $g(0, 0) = 0$. Let $(x_n, y_n)_{n \in \mathbf{N}}$ be any sequence in $X \times Y$ converging to zero and let W_0 be any full circled neighbourhood of zero in F. Choose an open neighbourhood W of zero satisfying $W + W \subset W_0$. For each $x \in X$, by Proposition 2.3, the mapping $g(x, \cdot)$ is continuous and hence the set $\{g(x, y_n): n \in \mathbf{N}\}$ is topologically bounded in F as $\{0\} \cup \{y_n: n \in \mathbf{N}\}$ is compact in Y. So if we put, for each $p \in \mathbf{N}$,

$$B_p = \{x \in X: g(x, y_n) \in 2^{p+1}W \text{ and } g(-x, y_n) \in 2^{p+1}W, \forall n \in \mathbf{N}\},$$

then $X = \bigcup_{p \in \mathbf{N}} B_p$ and hence, by Proposition 1.2, there exists an integer k and a circled neighbourhood V of zero in X with $V \subset \frac{1}{2}B_k - \frac{1}{2}B_k = \frac{1}{2}B_k + \frac{1}{2}B_k$. Therefore, for each $x = \frac{1}{2}b + \frac{1}{2}b' \in V$ with $b, b' \in B_k$ and each $n \in \mathbf{N}$, invoking the midpoint convexity of $g(\cdot, y_n)$ we have

$$g(x, y_n) \in \tfrac{1}{2}g(b, y_n) + \tfrac{1}{2}g(b', y_n) - P \subset 2^k W + 2^k W - P \subset 2^k W_0 - P$$

and hence again, by the midpoint convexity of $g(\cdot, y_n)$ and the relation $g(0, y_n) = 0$, we have

$$g(2^{-k}V, y_n) \subset (W_0 - P) \cap (W_0 + P) = W_0 \quad \text{for every } n \in \mathbf{N}.$$

As $\lim_{n\to\infty} x_n = 0$, we may conclude that $\lim_{n\to\infty} g(x_n, y_n) = 0$ and the proof is complete since $y \to f(c, d + y)$ is continuous. □

The above proof also gives the following result.

3.3. PROPOSITION. *Let f: $X \times Y \to F$ be a universally measurable midpoint P-convex-concave operator, that is $f(\cdot, y)$ and $-f(x, \cdot)$ are midpoint P-convex for each $(x, y) \in X \times Y$; then*:

(i) *f is continuous,*

(ii) *if P is closed, f is a continuous P-convex-concave operator.*

REMARKS. (1) If X and Y are also *separable* and if $\mathcal{C}(X)$ denotes the tribe of Christensen measurable subsets of X, that is (see **[4]**) the set of subsets $C \subset X$ for which there exist two universally measurable subsets A and M, a probability measure m on $\mathcal{U}(X)$ and a subset $N \subset M$ such that $C = A \cup N$ and $m(x + M) = 0$ for all $x \in X$, the above result still holds whenever f is separately $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ measurable.

(2) Results in the line of Proposition 3.2 about equicontinuous families of biconvex or concave-convex operators can be found in **[6]**.

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