CONTINUITY OF MEASURABLE CONVEX 
AND BICONVEX OPERATORS 

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Abstract. We prove that a mapping from the product of two complete metrizable 
vector spaces into a topological vector space which is separately universally measurable 
and separately convex with respect to a convex cone is continuous.

0. Introduction. If \( X \) is a complete separable vector space and if \( f \) is a midpoint \nconvex mapping from \( X \) into the real line \( \mathbb{R} \) which is Christensen measurable, then F. \nFischer and Z. Slodkowski have proved in [4] that \( f \) is continuous by showing that 
\( \{(x, r) \in X \times \mathbb{R} : f(x) < r\} \) is open in \( X \times \mathbb{R} \). Results of this kind about continuity 
of universally measurable morphisms were obtained before by A. Douady and L. \nSchwartz (see [7]) for linear operators between locally convex spaces, and by J. P. R. \nChristensen in [3] for homomorphisms between topological complete metrizable 
groups.

In this paper we are concerned with the study of convex operators taking values in 
a vector space. Vector analogues of the preceding results are proved. In fact, we 
establish that a separately universally measurable mapping \( f : X \times Y \to F \), where \( X \) 
and \( Y \) are two complete metrizable vector spaces and \( F \) is any topological vector 
space, is continuous whenever it is separately midpoint convex with respect to a 
convex cone.

Let us also note that many other interesting results about continuity of convex 
operators are given by J. M. Borwein in [1].

1. Preliminaries. Throughout this paper \( F \) will denote a (real separated) topologi-
cal vector space and \( X \) and \( Y \) two real complete metrizable topological vector spaces.

Let \( P \) be a convex cone in \( F \), i.e. \( tP + sP \subseteq P \) for all \( t, s \geq 0 \). One says that \( P \) is 
normal if there is a base of neighbourhoods \( V \) of zero with 
\[ V = (V + P) \cap (V - P). \]

Such neighbourhoods are said to be full. Many properties and examples of normal 
convex cones can be found in [7]. In the sequel we shall always assume that \( P \) is a 
normal convex cone in \( F \).

A subset \( B \) of a topological space \( S \) is universally measurable if for each finite 
measure \( m \) over the Borel tribe \( \mathcal{B}(S) \) the set \( B \) belongs to the \( m \)-completion of \( \mathcal{B}(S) \). 
The set of universally measurable subsets of \( S \) will be denoted by \( \mathcal{U}(S) \). 

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A mapping \( f: S \to F \) is universally measurable if for each open subset \( \Omega \) of \( F \) the set \( f^{-1}(\Omega) \) is in \( \mathcal{L}(S) \).

1.1. Remark. As a direct consequence of Proposition 8 in [5, p. 12], for each continuous mapping \( f \) from a topological space \( S \) into a topological space \( T \), one has \( f^{-1}(B) \in \mathcal{L}(S) \) for each \( B \in \mathcal{L}(T) \). □

Let us recall the following consequence of a very nice and important result of Christensen.

1.2. Proposition. If \( X = \bigcup_{n \in \mathbb{N}} B_n \) is a countable union of universally measurable subsets, then there exists an integer \( k \) such that \( B_k - B_k = \{ x - y : x, y \in B_k \} \) is a neighbourhood of zero.

Proof. This is a direct consequence of Theorem 7.1 in [3]. □

2. Midpoint convex operators.

2.1. Definition. One says that \( f: X \to F \) is \( P \)-convex if

\[
(2.1) \quad f(tx + (1 - t)y) \in tf(x) + (1 - t)f(y) - P \quad \text{for all } t \in [0,1],
\]

and \( f \) is midpoint \( P \)-convex if \( (2.1) \) holds for \( t = 1/2 \).

One observes that additive mappings are always midpoint convex.

Remark. It is easily shown that \( f \) is midpoint \( P \)-convex if and only if

\[
(2.2) \quad f(2^{-\kappa}kx + (1 - 2^{-\kappa})ky) \in 2^{-\kappa}kf(x) + (1 - 2^{-\kappa})kf(y) - P
\]

for all \( x, y \in X \), \( k \) and \( n \in \mathbb{N} \) with \( 0 \leq k \leq 2^n \). It follows that a midpoint \( P \) convex operator with closed epigraph is convex.

2.2. Lemma. A midpoint \( P \)-convex operator \( f: X \to F \) is continuous at \( a \in X \) if and only if \( f \) is upper semicontinuous at \( a \) in the following sense: for every neighbourhood \( W \) of zero in \( F \) there exists a neighbourhood \( V \) of zero in \( X \) such that

\[
(2.3) \quad f(a + x) \in f(a) + W - P \quad \text{for each } x \in V.
\]

Proof. It is clearly enough to show that the condition is sufficient. Let \( W \) be any neighbourhood of zero in \( F \). Choose a full circled neighbourhood \( W_0 \) of zero in \( Y \) with \( W_0 \subset W \) and a circled neighbourhood \( V \) of zero in \( X \) satisfying

\[
(2.3) \quad f(a + x) - f(a) \in W_0 - P \quad \text{for all } x \in V.
\]

As \( f \) is midpoint \( P \)-convex, we have for each \( x \in V \)

\[
f(a) \in \frac{1}{2}f(a - x) + \frac{1}{2}f(a + x) - P
\]

and hence

\[
f(a + x) - f(a) \in f(a) - f(a - x) + P \subset W_0 + P + P = W_0 + P.
\]

Making use of relation \( (2.3) \) once again we obtain

\[
f(a + x) - f(a) \in (W_0 - P) \cap (W_0 + P) = W_0 \subset W
\]

for each \( x \in V \). □

2.3. Proposition. Let \( f: X \to F \) be a universally measurable midpoint \( P \)-convex operator; then:

(i) \( f \) is continuous,

(ii) if \( P \) is closed, \( f \) is a continuous \( P \)-convex operator.
Proof. Let us begin by proving the continuity. Let \( a \) be any point in \( X \). Put \( g(x) = f(a + x) - f(a) \). The operator \( g \) is obviously a midpoint convex operator with \( g(0) = 0 \) and by Remark 1.1 it is universally measurable. Let \( W_0 \) be a neighbourhood of zero in \( F \). Choose an open circled neighbourhood \( W \) of zero with \( W + W \subset W_0 \). For each nonnegative integer \( n \) consider the universally measurable set

\[
B_n = \{ x \in X : g(x) \in 2^{n+1}W, g(-x) \in 2^{n+1}W \}.
\]

Since \( X = \bigcup_{n \in \mathbb{N}} B_n \), there exists, by Proposition 1.2, an integer \( k \) such that \( B_k - B_k \) is a neighbourhood of zero in \( X \), and for each \( x \) in the neighbourhood of zero \( V := \frac{1}{2}B_k - \frac{1}{2}B_k = \frac{1}{2}B_k + \frac{1}{2}B_k \) we may write \( x = \frac{1}{2}b + \frac{1}{2}b' \) with \( b, b' \in B_k \),

\[
g(x) = \frac{1}{2}g(b) + \frac{1}{2}g(b') - P \subset 2^kW + 2^kW - P \subset 2^kW_0 - P
\]

and hence

\[
2^{-k}g(x) \in W_0 - P.
\]

As \( g \) is midpoint \( P \)-convex with \( g(0) = 0 \), we have for each \( x \in V \),

\[
g(2^{-k}x) \in 2^{-k}g(x) - P \subset W_0 - P
\]

and hence by Lemma 2.2, \( g \) is continuous at 0, which implies that \( f \) is continuous.

If \( P \) is closed, then the continuity of \( f \) and relation (2.2) easily imply that \( f \) is \( P \)-convex. \( \square \)

3. Midpoint biconvex operators.

3.1. Definition. A mapping \( f : X \times Y \to F \) is called a midpoint \( P \)-biconvex operator if for each \( (x, y) \in X \times Y \) the mappings \( f(x, \cdot) \) and \( f(\cdot, y) \) are midpoint \( F \)-convex operators.

3.2. Proposition. Let \( f : X \times Y \to F \) be a separately universally measurable midpoint \( P \)-biconvex operator; then:

(i) \( f \) is continuous,

(ii) if \( P \) is closed, \( f \) is a continuous \( P \)-biconvex operator.

Proof. Let \((c, d)\) be any point in \( X \times Y \). Put \( g(x, y) = f(c + x, d + y) - f(c, d + y) \). The mapping \( g \) is separately universally measurable and \( g(0, 0) = 0 \). Let \((x_n, y_n)_{n \in \mathbb{N}} \) be any sequence in \( X \times Y \) converging to zero and let \( W_0 \) be any full circled neighbourhood of zero in \( F \). Choose an open neighbourhood \( W \) of zero satisfying \( W + W \subset W_0 \). For each \( x \in X \), by Proposition 2.3, the mapping \( g(x, \cdot) \) is continuous and hence the set \( \{ g(x, y_n) : n \in \mathbb{N} \} \) is topologically bounded in \( F \) as \( \{0\} \cup \{y_n : n \in \mathbb{N}\} \) is compact in \( Y \). So if we put, for each \( p \in \mathbb{N} \),

\[
B_p = \{ x \in X : g(x, y_n) \in 2^{p+1}W \text{ and } g(-x, y_n) \in 2^{p+1}W, \forall n \in \mathbb{N} \},
\]

then \( X = \bigcup_{p \in \mathbb{N}} B_p \) and hence, by Proposition 1.2, there exists an integer \( k \) and a circled neighbourhood \( V \) of zero in \( X \) with \( V \subset \frac{1}{2}B_k - \frac{1}{2}B_k = \frac{1}{2}B_k + \frac{1}{2}B_k \). Therefore, for each \( x = \frac{1}{2}b + \frac{1}{2}b' \in V \) with \( b, b' \in B_k \) and each \( n \in \mathbb{N} \), invoking the midpoint convexity of \( g(\cdot, y_n) \) we have

\[
g(x, y_n) \in \frac{1}{2}g(b, y_n) + \frac{1}{2}g(b', y_n) - P \subset 2^kW + 2^kW - P \subset 2^kW_0 - P
\]
and hence again, by the midpoint convexity of \( g(\cdot, y_n) \) and the relation \( g(0, y_n) = 0 \), we have
\[
g(2^{-k}V, y_n) \subseteq (W_0 - P) \cap (W_0 + P) = W_0 \quad \text{for every } n \in \mathbb{N}.
\]
As \( \lim_{n \to \infty} x_n = 0 \), we may conclude that \( \lim_{n \to \infty} g(x_n, y_n) = 0 \) and the proof is complete since \( y \to f(c, d + y) \) is continuous. □

The above proof also gives the following result.

3.3. **Proposition.** Let \( f: X \times Y \to F \) be a universally measurable midpoint \( P \)-convex-concave operator, that is \( f(\cdot, y) \) and \( -f(x, \cdot) \) are midpoint \( P \)-convex for each \((x, y) \in X \times Y\); then:

(i) \( f \) is continuous,

(ii) if \( P \) is closed, \( f \) is a continuous \( P \)-convex-concave operator.

**Remarks.** (1) If \( X \) and \( Y \) are also separable and if \( \mathcal{C}(X) \) denotes the tribe of Christensen measurable subsets of \( X \), that is (see [4]) the set of subsets \( C \subseteq X \) for which there exist two universally measurable subsets \( A \) and \( M \), a probability measure \( m \) on \( \mathcal{B}(X) \) and a subset \( N \subseteq M \) such that \( C = A \cup N \) and \( m(x + M) = 0 \) for all \( x \in X \), the above result still holds whenever \( f \) is separately \( \mathcal{C}(X) \) and \( \mathcal{C}(Y) \) measurable.

(2) Results in the line of Proposition 3.2 about equicontinuous families of biconvex or concave-convex operators can be found in [6].

**References**