

SURJECTIVITY OF ϕ -ACCRETIVE OPERATORS

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ABSTRACT. Let X and Y be Banach spaces, $\phi: X \rightarrow Y^*$ and $P: X \rightarrow Y$; P is said to be strongly ϕ -accretive if $\langle Px - Py, \phi(x - y) \rangle \geq c\|x - y\|^2$ for some $c > 0$ and each $x, y \in X$. These maps constitute a generalization simultaneously of monotone maps (when $Y = X^*$) and accretive maps (when $Y = X$). By applying the Caristi-Kirk fixed point theorem, W. O. Ray showed that a localized class of these maps must be surjective under appropriate geometric assumptions on Y^* and continuity assumptions on the duality map. In this paper we show that such geometric assumptions can be removed without affecting the conclusion of Ray.

Let X and Y be Banach spaces with Y^* the dual of Y , and let $\phi: X \rightarrow Y^*$ be a map satisfying:

(1) $\phi(X)$ is dense in Y^*

(2) for each $x \in X$ and each $\alpha \geq 0$, $\|\phi(x)\| \leq \|x\|$ and $\phi(\alpha x) = \alpha\phi(x)$.

A map $P: X \rightarrow Y$ is said to be strongly ϕ -accretive [1] if there exists a constant $c > 0$ such that, for all $u, v \in X$,

(3)
$$\langle Pu - Pv, \phi(u - v) \rangle \geq c\|u - v\|^2.$$

The ϕ -accretive maps were introduced in an effort to unify the theories for monotone maps (when $Y = X^*$) and for accretive maps (when $Y = X$). Those maps were studied by F. E. Browder [1-4], W. A. Kirk [7], and W. O. Ray [8]. In fact, Browder obtained the following.

THEOREM (F. E. BROWDER [4]). *Let X and Y be Banach spaces and $P: X \rightarrow Y$ strongly ϕ -accretive. If Y^* is uniformly convex and P is locally Lipschitzian, then $P(X) = Y$.*

A map $P: X \rightarrow Y$ is said to be locally strongly ϕ -accretive [7] if, for each $y \in Y$ and $r > 0$, there exists a constant $c > 0$ such that

(4) if $\|Px - y\| \leq r$, then, for all $u \in X$ sufficiently near to x ,

$$\langle Pu - Px, \phi(u - x) \rangle \geq c\|x - u\|^2.$$

Kirk [7] extended the surjectivity theorem of Browder to the class of locally strongly ϕ -accretive maps by applying the Caristi-Kirk fixed point theorem [5, 6, 7].

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For a Banach space Y , we denote by J the duality map from Y into 2^{Y^*} given by

$$J(y) = \{y^* \in Y^* \mid \|y^*\|^2 = \|y\|^2 = \langle y, y^* \rangle\}.$$

It is well known that, by the Hahn-Banach theorem, $J(y)$ is not empty for each $y \in Y$, J is single valued whenever Y^* is strictly convex, and J is uniformly continuous on bounded subsets of Y whenever Y^* is uniformly convex.

Recently, Ray [8] extended Browder's theorem as follows:

THEOREM (W. O. RAY [8]). *Let X and Y be Banach spaces and $P: X \rightarrow Y$ a locally Lipschitzian and locally strongly ϕ -accretive map. If Y^* is strictly convex and J is continuous, and if $P(X)$ is closed in Y , then $P(X) = Y$.*

In this paper we show that the strict convexity of Y^* can be removed without affecting the conclusion.

The duality map J is said to be strongly upper semicontinuous if the following condition holds:

(5) if $\lim_{n \rightarrow \infty} y_n = y$, $y_n^* \in J(y_n)$, and $y^* \in J(y)$, then y^* is a subsequential (strong) limit of $\{y_n^*\}$.

Note that a continuous single-valued map is strongly u.s.c., not conversely, and that a strongly u.s.c. multivalued map is u.s.c., not conversely.

An example of a strongly u.s.c. single valued map F which is not continuous is as follows: Let f be a bijection from the set N of natural numbers onto the set of rational numbers in $[0, 1]$, $X = \{0\} \cup \{1/n \mid n \in N\}$, and $Y = \{f(n) \mid n \in N\}$. Let $F: X \rightarrow Y$ be a map such that $F(1/n) = f(n)$ and $F(0) = 1/2$. Then F is not continuous. However, F is strongly u.s.c., for we can find a subsequence $\{f(n_k)\}$ converging to $F(0) = 1/2$.

Now, we state our main results.

THEOREM 1. *Let X and Y be Banach spaces, and $P: X \rightarrow Y$ a locally Lipschitzian and strongly ϕ -accretive map. If the duality map J of Y is strongly upper semicontinuous, then $P(X) = Y$.*

THEOREM 2. *Let X and Y be Banach spaces and $P: X \rightarrow Y$ a locally Lipschitzian and locally strongly ϕ -accretive map. If the duality map J of Y is strongly upper semicontinuous and $P(X)$ is closed, then $P(X) = Y$.*

We note that if P is strongly ϕ -accretive, then $P(X)$ is closed in Y . Therefore, Theorem 1 is a consequence of Theorem 2.

LEMMA. *For any $y \in Y$, $y^* \in J(y)$, and $\epsilon > 0$, there exists an $h \in X$ such that $\|h\| \geq 1$ and $\|\phi(h) - y^*\| < \epsilon$.*

PROOF. It suffices to show that for any $f \in Y^*$ with $\|f\| = 1$ there exists $h \in X$ such that $\|h\| \geq 1$ and $\|\phi(h) - f\| < \epsilon$. Suppose there exist $f \in Y^*$ and $\epsilon_0 > 0$ such that $\|f\| = 1$ and, for any $h \in X$ with $\|h\| \geq 1$, $\|\phi(h) - f\| \geq \epsilon_0$. Since $\phi(X)$ is dense

in Y^* , there exist $x_n \in X$ such that $\|\phi(x_n) - f\| < 1/n$ and, by the above assumption, $\|x_n\| < 1$ for all sufficiently large n . Hence $\|f\| - \|\phi(x_n)\| < 1/n$ or $\|f\| - 1/n < \|\phi(x_n)\|$, and, by property (2) of ϕ , we have $1 - 1/n < \|\phi(x_n)\| \leq \|x_n\| < 1$ for all sufficiently large n . Therefore,

$$\lim_{n \rightarrow \infty} \|\phi(x_n)\| = \lim_{n \rightarrow \infty} \|x_n\| = 1.$$

But $\|x_n\|^{-1}\phi(x_n) = \phi(x_n\|x_n\|^{-1})$ by (2) and $\|x_n\|x_n\|^{-1}\| = 1$. Hence we have $\|\phi(x_n)\|x_n\|^{-1} - f\| \geq \epsilon_0 > 0$ from the assumption. Since

$$\begin{aligned} \|\phi(x_n)/\|x_n\| - f\| &\leq \|f - \phi(x_n)\| + \|\phi(x_n) - \phi(x_n)/\|x_n\|\| \\ &< 1/n + |1 - 1/\|x_n\|| \|\phi(x_n)\| \end{aligned}$$

for all sufficiently large n , we have $\lim_{n \rightarrow \infty} \|\phi(x_n)\|x_n\|^{-1} - f\| = 0$, a contradiction.

Note that for any $y^* \in J(y) \in 2^{Y^*}$ we have

$$(5) \quad \|y\|^2 \leq \|z\|^2 - 2\langle z - y, y^* \rangle \quad \text{for any } z \in Y.$$

Indeed,

$$\begin{aligned} \|z\|^2 - 2\langle z - y, y^* \rangle - \|y\|^2 &= \|z\|^2 - 2\langle z, y^* \rangle + 2\langle y, y^* \rangle - \|y\|^2 \\ &= \|z\|^2 - 2\langle z, y^* \rangle + \|y\|^2 \geq \|z\|^2 - 2\|z\| \|y\| + \|y\|^2 \\ &= (\|y\| - \|z\|)^2 \geq 0. \end{aligned}$$

We use the notation

$$B(x, r) = \{w \in E \mid \|w - x\| \leq r\}$$

for $E = X$ or $E = Y$.

PROOF OF THEOREM 2. $P(X)$ is open: Since $J(y) \neq \emptyset$ for each $y \in Y$, we can choose $y^* \in J(y)$. For a given $x_0 \in X$, choose $\epsilon_1 > 0$ so small that P is Lipschitzian with constant M on $B(x_0, 2\epsilon_1)$. Choose $c > 0$ and $\epsilon_2 > 0$ so that (4) holds on $B(Px_0; 2M\epsilon_1)$ whenever $\|u - x_0\| \leq 2\epsilon_2$; set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and set $r = \min\{c\epsilon/2, M\epsilon\}$. Now it suffices to show that $B(Px_0, r) \subset P(X)$. Suppose $y \in B(Px_0, r)$ and $y \notin P(X)$. Then $d = \text{dist}(y, P(X)) > 0$. Let $D = \{x \in B(x_0, \epsilon) \mid \|y - Px\| \leq r\}$. Since $x_0 \in D$, D is a nonempty closed subset of X , and hence D is complete. For any $x \in D$, by the Lemma there exists $h \in X$ such that $\|h\| \geq 1$ and

$$(6) \quad \|\phi(h) - (y - Px)^*\|y - Px\|^{-1}\| \leq c/2M.$$

Set $x_t = x + th, t > 0$. By (4), for t sufficiently small we have

$$\langle Px_t - Px, \phi(x_t - x) \rangle \geq c\|x_t - x\|^2$$

or

$$\langle Px_t - Px, \phi(h) \rangle \geq ct\|h\|^2.$$

If t is so small that $\|x_t - x\| \leq \epsilon$, and hence $x_t \in B(x_0, 2\epsilon_1)$, then

$$\langle Px_t - Px, \phi(h) \rangle \geq (c/M)\|Px_t - Px\|.$$

Indeed, $\|Px_t - Px\| \leq M\|x_t - x\|$ for $x, x_t \in B(x_0, 2\varepsilon_1)$. By applying (6),

$$\begin{aligned}
 (7) \quad & \langle Px_t - Px, (y - Px)^* \rangle \\
 & = \langle Px_t - Px, \|y - Px\|\phi(h) - \|y - Px\|\phi(h) + (y - Px)^* \rangle \\
 & \geq (c/M)\|Px_t - Px\| \|y - Px\| - (c/2M)\|Px_t - Px\| \|y - Px\| \\
 & \geq (c/2M)\|y - Px\| \|Px_t - Px\|.
 \end{aligned}$$

From (5) and (7), we have

$$\begin{aligned}
 \|y - Px_t\|^2 & \leq \|y - Px\|^2 - 2\langle Px_t - Px, (y - Px_t)^* \rangle \\
 & \leq \|y - Px\|^2 - (cd/M)\|Px_t - Px\| \\
 & \quad + 2\|Px_t - Px\| \|(y - Px)^* - (y - Px_t)^*\|.
 \end{aligned}$$

Since $y - Px_t \rightarrow y - Px$ as $t \rightarrow 0$ and J is strongly u.s.c., we may select $t > 0$ so small that $\|(y - Px)^* - (y - Px_t)^*\| \leq cd/4M$. Therefore,

$$\|y - Px_t\|^2 \leq \|y - Px\|^2 - (cd/2M)\|Px_t - Px\|,$$

that is

$$(cd/2M)\|x_t - x\| \leq \|y - Px\|^2 - \|y - Px_t\|^2.$$

Hence $\|y - Px_t\| \leq \|y - Px\| \leq r$ and $x_t \in B(x_0, 2\varepsilon)$. Indeed, $x_t \in B(x_0, 2\varepsilon)$ and $\|y - Px_t\| \leq r$ imply

$$\begin{aligned}
 \|x_t - x_0\| & \leq c^{-1}\|Px_t - Px_0\| \\
 & \leq c^{-1}(\|Px_t - y\| + \|y - Px_0\|) \leq 2rc^{-1} \leq \varepsilon.
 \end{aligned}$$

Let $g: D \rightarrow D$ such that $gx = x_t \in D$, and let $\psi(x) = (2M/c^2d)\|y - Px\|^2$. Then $\|x - gx\| \leq \psi(x) - \psi(gx)$. Since ψ is the continuous map from the complete metric space D into the nonnegative reals, by the Caristi-Kirk fixed point theorem, g has a fixed point in D . Since $\|x_t - x\| = t\|h\| \neq 0$, this is a contradiction.

Finally, note that Theorems 1 and 2 are generalizations of the surjectivity theorems of Browder and Ray, respectively. Note also that the geometric structures of Y^* in their results are not necessary.

REFERENCES

1. F.E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc. Sympos. Pure Math., Vol. 18, Amer. Math. Soc., Providence, R.I., 1976.
2. ———, *Normal solvability and existence theorems for nonlinear mappings in Banach spaces*, Problems in Nonlinear Analysis, Edizioni Cremonese, Rome, Italy, 1971, pp. 19–35.
3. ———, *Normal solvability for nonlinear mappings and the geometry of Banach spaces*, Problems in Nonlinear Analysis, Edizioni Cremonese, Rome, Italy, 1971, pp. 37–66.
4. ———, *Normal solvability and ϕ -accretive mappings of Banach spaces*, Bull. Amer. Math. Soc. **78** (1972), 186–192.
5. J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251.
6. I. Ekeland, *Sur les problèmes variationnels*, C. R. Acad. Sci. Paris **275** (1972), 1057–1059.
7. W. A. Kirk, *Caristi's fixed point theorem and the theory of normal solvability*, Proc. Conf. Fixed Point Theory and its Applications (Dalhousie Univ., June 1975), Academic Press, 1976, pp. 109–120.
8. W. O. Ray, *Phi-accretive operators and Ekeland's theorem*, J. Math. Anal. Appl. **88** (1982), 566–571.

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