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SURJECTIVITY OF ϕ-ACCRETIVE OPERATORS

JONG AN PARK AND SEHIE PARK[1]

ABSTRACT. Let X and Y be Banach spaces, ϕ: $X \to Y^*$ and P: $X \to Y$; P is said to be strongly ϕ-accretive if $\langle Px - Py, \phi(x-y)\rangle \geqslant c\|x-y\|^2$ for some $c > 0$ and each $x, y \in X$. These maps constitute a generalization simultaneously of monotone maps (when $Y = X^*$) and accretive maps (when $Y = X$). By applying the Caristi-Kirk fixed point theorem, W. O. Ray showed that a localized class of these maps must be surjective under appropriate geometric assumptions on Y^* and continuity assumptions on the duality map. In this paper we show that such geometric assumptions can be removed without affecting the conclusion of Ray.

Let X and Y be Banach spaces with Y^* the dual of Y, and let ϕ: $X \to Y^*$ be a map satisfying:

(1) $\phi(X)$ is dense in Y^*

(2) for each $x \in X$ and each $\alpha \geqslant 0$, $\|\phi(x)\| \leqslant \|x\|$ and $\phi(\alpha x) = \alpha\phi(x)$.

A map P: $X \to Y$ is said to be strongly ϕ-accretive [**1**] if there exists a constant $c > 0$ such that, for all $u, v \in X$,

$$\langle Pu - Pv, \phi(u-v)\rangle \geqslant c\|u-v\|^2. \tag{3}$$

The ϕ-accretive maps were introduced in an effort to unify the theories for monotone maps (when $Y = X^*$) and for accretive maps (when $Y = X$). Those maps were studied by F. E. Browder [**1–4**], W. A. Kirk [**7**], and W. O. Ray [**8**]. In fact, Browder obtained the following.

THEOREM (F. E. BROWDER [**4**]). *Let X and Y be Banach spaces and P: $X \to Y$ strongly ϕ-accretive. If Y^* is uniformly convex and P is locally Lipschitzian, then $P(X) = Y$.*

A map P: $X \to Y$ is said to be locally strongly ϕ-accretive [**7**] if, for each $y \in Y$ and $r > 0$, there exists a constant $c > 0$ such that

(4) if $\|Px - y\| \leqslant r$, then, for all $u \in X$ sufficiently near to x,

$$\langle Pu - Px, \phi(u-x)\rangle \geqslant c\|x-u\|^2.$$

Kirk [**7**] extended the surjectivity theorem of Browder to the class of locally strongly ϕ-accretive maps by applying the Caristi-Kirk fixed point theorem [**5, 6, 7**].

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For a Banach space Y, we denote by J the duality map from Y into 2^{Y^*} given by

$$J(y) = \left\{ y^* \in Y^* \mid \|y^*\|^2 = \|y\|^2 = \langle y, y^* \rangle \right\}.$$

It is well known that, by the Hahn-Banach theorem, $J(y)$ is not empty for each $y \in Y$, J is single valued whenever Y^* is strictly convex, and J is uniformly continuous on bounded subsets of Y whenever Y^* is uniformly convex.

Recently, Ray **[8]** extended Browder's theorem as follows:

THEOREM (W. O. RAY **[8]**). *Let X and Y be Banach spaces and P: $X \to Y$ a locally Lipschitzian and locally strongly ϕ-accretive map. If Y^* is strictly convex and J is continuous, and if $P(X)$ is closed in Y, then $P(X) = Y$.*

In this paper we show that the strict convexity of Y^* can be removed without affecting the conclusion.

The duality map J is said to be strongly upper semicontinuous if the following condition holds:

(5) if $\lim_{n\to\infty} y_n = y$, $y_n^* \in J(y_n)$, and $y^* \in J(y)$, then y^* is a subsequential (strong) limit of $\{y_n^*\}$.

Note that a continuous single-valued map is strongly u.s.c., not conversely, and that a strongly u.s.c. multivalued map is u.s.c., not conversely.

An example of a strongly u.s.c. single valued map F which is not continuous is as follows: Let f be a bijection from the set N of natural numbers onto the set of rational numbers in $[0, 1]$, $X = \{0\} \cup \{1/n \mid n \in N\}$, and $Y = \{f(n) \mid n \in N\}$. Let F: $X \to Y$ be a map such that $F(1/n) = f(n)$ and $F(0) = 1/2$. Then F is not continuous. However, F is strongly u.s.c., for we can find a subsequence $\{f(n_k)\}$ converging to $F(0) = 1/2$.

Now, we state our main results.

THEOREM 1. *Let X and Y be Banach spaces, and P: $X \to Y$ a locally Lipschitzian and strongly ϕ-accretive map. If the duality map J of Y is strongly upper semicontinuous, then $P(X) = Y$.*

THEOREM 2. *Let X and Y be Banach spaces and P: $X \to Y$ a locally Lipschitzian and locally strongly ϕ-accretive map. If the duality map J of Y is strongly upper semicontinuous and $P(X)$ is closed, then $P(X) = Y$.*

We note that if P is strongly ϕ-accretive, then $P(X)$ is closed in Y. Therefore, Theorem 1 is a consequence of Theorem 2.

LEMMA. *For any $y \in Y$, $y^* \in J(y)$, and $\varepsilon > 0$, there exists an $h \in X$ such that $\|h\| \geqslant 1$ and $\|\phi(h) - y^*\|y\|^{-1}\| < \varepsilon$.*

PROOF. It suffices to show that for any $f \in Y^*$ with $\|f\| = 1$ there exists $h \in X$ such that $\|h\| \geqslant 1$ and $\|\phi(h) - f\| < \varepsilon$. Suppose there exist $f \in Y^*$ and $\varepsilon_0 > 0$ such that $\|f\| = 1$ and, for any $h \in X$ with $\|h\| \geqslant 1$, $\|\phi(h) - f\| \geqslant \varepsilon_0$. Since $\phi(X)$ is dense in Y^*, there exist $x_n \in X$ such that $\|\phi(x_n) - f\| < 1/n$ and, by the above assumption, $\|x_n\| < 1$ for all sufficiently large n. Hence $\|f\| - \|\phi(x_n)\| < 1/n$ or $\|f\| - 1/n < \|\phi(x_n)\|$, and, by property (2) of ϕ, we have $1 - 1/n < \|\phi(x_n)\| \leqslant \|x_n\| < 1$ for all sufficiently large n. Therefore,

$$\lim_{n\to\infty} \|\phi(x_n)\| = \lim_{n\to\infty} \|x_n\| = 1.$$

But $\|x_n\|^{-1}\phi(x_n) = \phi(x_n\|x_n\|^{-1})$ by (2) and $\|x_n\|x_n\|^{-1}\| = 1$. Hence we have $\|\phi(x_n)\|x_n\|^{-1} - f\| \geqslant \varepsilon_0 > 0$ from the assumption. Since

$$\begin{aligned}\|\phi(x_n)/\|x_n\| - f\| &\leqslant \|f - \phi(x_n)\| + \|\phi(x_n) - \phi(x_n)/\|x_n\|\| \\ &< 1/n + |1 - 1/\|x_n\|| \, \|\phi(x_n)\|\end{aligned}$$

for all sufficiently large n, we have $\lim_{n\to\infty} \|\phi(x_n)\|x_n\|^{-1} - f\| = 0$, a contradiction.

Note that for any $y^* \in J(y) \in 2^{Y^*}$ we have

$$\|y\|^2 \leqslant \|z\|^2 - 2\langle z - y, y^*\rangle \quad \text{for any } z \in Y. \tag{5}$$

Indeed,

$$\begin{aligned}\|z\|^2 - 2\langle z - y, y^*\rangle - \|y\|^2 &= \|z\|^2 - 2\langle z, y^*\rangle + 2\langle y, y^*\rangle - \|y\|^2 \\ &= \|z\|^2 - 2\langle z, y^*\rangle + \|y\|^2 \geqslant \|z\|^2 - 2\|z\| \, \|y\| + \|y\|^2 \\ &= (\|y\| - \|z\|)^2 \geqslant 0.\end{aligned}$$

We use the notation

$$B(x, r) = \{w \in E \mid \|w - x\| \leqslant r\}$$

for $E = X$ or $E = Y$.

Proof of Theorem 2. $P(X)$ is open: Since $J(y) \neq \phi$ for each $y \in Y$, we can choose $y^* \in J(y)$. For a given $x_0 \in X$, choose $\varepsilon_1 > 0$ so small that P is Lipschitzian with constant M on $B(x_0, 2\varepsilon_1)$. Choose $c > 0$ and $\varepsilon_2 > 0$ so that (4) holds on $B(Px_0; 2M\varepsilon_1)$ whenever $\|u - x_0\| \leqslant 2\varepsilon_2$; set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and set $r = \min\{c\varepsilon/2, M\varepsilon\}$. Now it suffices to show that $B(Px_0, r) \subset P(X)$. Suppose $y \in B(Px_0, r)$ and $y \notin P(X)$. Then $d = \operatorname{dist}(y, P(X)) > 0$. Let $D = \{x \in B(x_0, \varepsilon) \mid \|y - Px\| \leqslant r\}$. Since $x_0 \in D$, D is a nomempty closed subset of X, and hence D is complete. For any $x \in D$, by the Lemma there exists $h \in X$ such that $\|h\| \geqslant 1$ and

$$\|\phi(h) - (y - Px)^*\|y - Px\|^{-1}\| \leqslant c/2M. \tag{6}$$

Set $x_t = x + th$, $t > 0$. By (4), for t sufficiently small we have

$$\langle Px_t - Px, \phi(x_t - x)\rangle \geqslant c\|x_t - x\|^2$$

or

$$\langle Px_t - Px, \phi(h)\rangle \geqslant ct\|h\|^2.$$

If t is so small that $\|x_t - x\| \leqslant \varepsilon$, and hence $x_t \in B(x_0, 2\varepsilon_1)$, then

$$\langle Px_t - Px, \phi(h)\rangle \geqslant (c/M)\|Px_t - Px\|.$$

Indeed, $\|Px_t - Px\| \leqslant M\|x_t - x\|$ for $x, x_t \in B(x_0, 2\varepsilon_1)$. By applying (6),

$$\begin{aligned} &\langle Px_t - Px, (y - Px)^* \rangle \\ (7) \qquad &= \langle Px_t - Px, \|y - Px\|\phi(h) - \|y - Px\|\phi(h) + (y - Px)^* \rangle \\ &\geqslant (c/M)\|Px_t - Px\|\, \|y - Px\| - (c/2M)\|Px_t - Px\|\, \|y - Px\| \\ &\geqslant (c/2M)\|y - Px\|\, \|Px_t - Px\|. \end{aligned}$$

From (5) and (7), we have

$$\begin{aligned} \|y - Px_t\|^2 &\leqslant \|y - Px\|^2 - 2\langle Px_t - Px, (y - Px_t)^* \rangle \\ &\leqslant \|y - Px\|^2 - (cd/M)\|Px_t - Px\| \\ &\quad + 2\|Px_t - Px\|\, \|(y - Px)^* - (y - Px_t)^*\|. \end{aligned}$$

Since $y - Px_t \to y - Px$ as $t \to 0$ and J is strongly u.s.c., we may select $t > 0$ so small that $\|(y - Px)^* - (y - Px_t)^*\| \leqslant cd/4M$. Therefore,

$$\|y - Px_t\|^2 \leqslant \|y - Px\|^2 - (cd/2M)\|Px_t - Px\|,$$

that is

$$(cd/2M)\|x_t - x\| \leqslant \|y - Px\|^2 - \|y - Px_t\|^2.$$

Hence $\|y - Px_t\| \leqslant \|y - Px\| \leqslant r$ and $x_t \in B(x_0, 2\varepsilon)$. Indeed, $x_t \in B(x_0, 2\varepsilon)$ and $\|y - Px_t\| \leqslant r$ imply

$$\begin{aligned} \|x_t - x_0\| &\leqslant c^{-1}\|Px_t - Px_0\| \\ &\leqslant c^{-1}\left(\|Px_t - y\| + \|y - Px_0\|\right) \leqslant 2rc^{-1} \leqslant \varepsilon. \end{aligned}$$

Let $g\colon D \to D$ such that $gx = x_t \in D$, and let $\psi(x) = (2M/c^2d)\|y - Px\|^2$. Then $\|x - gx\| \leqslant \psi(x) - \psi(gx)$. Since ψ is the continuous map from the complete metric space D into the nonnegative reals, by the Caristi-Kirk fixed point theorem, g has a fixed point in D. Since $\|x_t - x\| = t\|h\| \neq 0$, this is a contradiction.

Finally, note that Theorems 1 and 2 are generalizations of the surjectivity theorems of Browder and Ray, respectively. Note also that the geometric structures of Y^* in their results are not necessary.

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Department of Mathematics, Seoul National University, Seoul 151, Korea