

ANALYTIC FUNCTIONS WITH LARGE SETS OF FATOU POINTS

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ABSTRACT. For a function f analytic in the unit disc D , and for each $\lambda > 0$, let $L(\lambda) = \{z \in D: |f(z)| = \lambda\}$ denote a level set for f . We introduce a class \mathcal{L}_1 of functions characterized by geometric properties of a collection of sets $\{L(\lambda_n)\}$, where $\{\lambda_n\}$ is an unbounded sequence. We show that \mathcal{L}_1 is a proper subclass of the class \mathcal{L} of G. R. MacLane. Let A_∞ denote the set of points $e^{i\theta}$ at which the function f has ∞ as an asymptotic value, and let $F(f)$ denote the set of Fatou points of f . We prove that for a function f in the class \mathcal{L}_1 , if Γ is an arc of the unit circle such that $\Gamma \cap A_\infty = \emptyset$, then almost every point of Γ belongs to $F(f)$.

1. Introduction. Let $D = \{z: |z| < 1\}$, $C = \{z: |z| = 1\}$, and let f be a function analytic in D . For each real number $\lambda > 0$, let $L(\lambda) = \{z \in D: |f(z)| = \lambda\}$ be a level set for f . Let $F(f)$ denote the set of Fatou points of f , that is,

$$F(f) = \{e^{i\theta} \in C: f \text{ has an angular limit at } e^{i\theta}\},$$

and let

$$A_\infty = \{e^{i\theta} \in C: f \text{ has } \infty \text{ as an asymptotic value at } e^{i\theta}\}.$$

Finally, if E is a subset of the closure of D , let $|E|$ denote the linear measure of E . In particular, if γ is a curve in $D \cup C$ then $|\gamma|$ is the arc length of the curve γ .

We use the following theorem of S. Dragosh [6, Theorem 3, p. 9] as our point of departure.

THEOREM D. *There exists a function f analytic in D such that for each arc Γ of C both $\Gamma \cap A_\infty = \emptyset$ and $0 < |\Gamma \cap F(f)| < |\Gamma|$.*

(We note that G. R. MacLane [7, Example 11, p. 75] obtained a slightly weaker result.)

In this paper, we will study the following question.

Question. *What are sufficient conditions on a function f , analytic in D to guarantee that*

(S) *if Γ is an arc of C with $\Gamma \cap A_\infty = \emptyset$ then $|F(f) \cap \Gamma| = |\Gamma|$?*

The hypothesis that $\Gamma \cap A_\infty = \emptyset$ is made to exclude, for example, the modular function $M(z)$.

We introduce the class of functions \mathcal{L}_1 and we show that \mathcal{L}_1 is a subclass of the class \mathcal{L} of G. R. MacLane [7]. (The class \mathcal{L} is the collection of all functions analytic in

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D such that each of the sets $L(\lambda)$ ends at points.) In our main result (Theorem 2), we show that if $f \in \mathcal{L}_1$ then (S) is satisfied.

To define the class \mathcal{L}_1 we need some additional notation. Throughout, the letters n, m, j and k are positive integers, and the letter i is one of the letters “ l ” or “ u ”. For $\lambda > 0$, let $D^l(\lambda) = \{z \in D: |f(z)| < \lambda\}$ and let $D^u(\lambda) = \{z \in D: |f(z)| > \lambda\}$. The notations $D_n^l(\lambda)$ and $D_n^u(\lambda)$ will be used to denote components of $D^l(\lambda)$ and $D^u(\lambda)$, respectively, and we will write $D^i(\lambda) = \cup_n D_n^i(\lambda)$ for $i \in \{l, u\}$. For a fixed n and a fixed $i \in \{l, u\}$, the boundary, $\partial D_n^i(\lambda)$, of $D_n^i(\lambda)$ consists of a union of arcs in $L(\lambda)$ together with the set $\gamma_n^i(\lambda) = \partial D_n^i(\lambda) \cap C$ (which could be the empty set). Let $(D_n^i(\lambda))^*$ be the smallest simply connected region containing $D_n^i(\lambda)$. We note that each $D_n^i(\lambda)$ is already simply connected, so $D_n^i(\lambda) = (D_n^i(\lambda))^*$. The region $(D_n^u(\lambda))^*$ is simply the region $D_n^u(\lambda)$ together with all the “holes” in this region.

We say that the number $\lambda > 0$ is an *admissible value* for f if the following are satisfied:

- (1) If $\gamma_n^i(\lambda) \neq \emptyset$, then $|\partial(D_n^i(\lambda))^*| < \infty$, where $i \in \{l, u\}$.
- (2) If $\{n_j\}$ is a sequence of positive integers such that $n_j \rightarrow \infty$ and $\gamma_{n_j}^i(\lambda) = \emptyset$ for each j , then $\text{diam } D_{n_j}^i(\lambda) \rightarrow 0$.
- (3) $\sum_n |\gamma_n^l(\lambda)| + \sum_n |\gamma_n^u(\lambda)| = 2\pi$.

We say that f is in the class \mathcal{L}_1 if there exists a sequence $\{\lambda_k\}$ of admissible values for f such that $\lambda_k \rightarrow \infty$.

In §2, we prove that \mathcal{L}_1 is a proper subclass of MacLane’s class \mathcal{L} , and also that if f is in the class \mathcal{L}_1 then (S) is satisfied. Thus, the class \mathcal{L}_1 gives an answer to the question we raised.

In §3, we give an example to show that the condition that f be in the class \mathcal{L}_1 is not a necessary condition to give an affirmative answer to our question. In addition, we give an example of a function f in the class \mathcal{L}_1 for which $A_\infty = C$ and $|F(f)| = 0$. Finally, we show that no containment relationship exists between the class \mathcal{L}_1 and the class of normal functions.

In §4, we give some open questions.

2. The main results. We begin with some lemmas concerning the geometry of the sets $D_n^i(\lambda)$, where $i \in \{l, u\}$ and λ is an admissible value.

We will use several times a result due to F. and M. Riesz (see [5, Theorem 3.3, p. 50]).

THEOREM R. *Let φ be a one-to-one conformal mapping from D onto a simply connected region R bounded by a rectifiable closed curve Γ . Then φ can be extended continuously to the closure of D , $\varphi'(e^{i\theta})$ exists for almost all $\theta \in [0, 2\pi)$, and*

$$|\Gamma| = \int_0^{2\pi} |\varphi'(e^{i\theta})| d\theta.$$

In addition, if E is a subset of Γ then $|E| = 0$ if and only if $|\varphi^{-1}(E)| = 0$.

This result is normally stated for the case where Γ is a Jordan curve. However, the proof appearing in [5, pp. 50–53] is valid for the case where Γ is a closed rectifiable curve (and not necessarily a simple closed curve).

LEMMA 1. Let f be a function analytic in D , let $i \in \{l, u\}$, and let $\lambda > 0$ be an admissible value for f . For each n and for each point $p \in \gamma_n^i(\lambda)$, there exists a Jordan arc γ_p in $D_n^i(\lambda) \cup \{p\}$ such that p is an endpoint of γ_p .

PROOF. Let $p \in \gamma_n^i(\lambda)$. Because of (1), $\partial(D_n^i(\lambda))^*$ is a rectifiable closed curve. Let φ be a conformal mapping from D onto $(D_n^i(\lambda))^*$. By Theorem R, φ can be extended continuously to the closure of D . Thus, there exists a radius R_p of D such that $\varphi(R_p) = \gamma_p'$ is an arc in $(D_n^i(\lambda))^*$ ending at p .

If $\gamma_p' \subset D_n^i(\lambda) \cup \{p\}$, then $\gamma_p = \gamma_p'$ is the desired curve. However, if γ_p' meets one or more "holes" in $D_n^i(\lambda)$, then we can construct a sequence of detours around these "holes". Further, because of (2), the diameters of these detours will be small for the "holes" which are near the unit circle C . Thus, there exists Jordan arc γ_p' obtained from γ_p' by means of appropriate detours around the "holes" in $D_n^i(\lambda)$, so that $\gamma_p \subset D_n^i(\lambda) \cup \{p\}$ and γ_p has p as an endpoint. (The formal construction of these detours is a standard elementary exercise, so we omit the details.)

LEMMA 2. Let f be a function analytic in D and let $\lambda > 0$ be an admissible value for f . Let $E(\lambda) = \{p \in C: p \in \gamma_n^i(\lambda) \cap \gamma_m^{i'}(\lambda), \text{ where either } i \neq i' \text{ or } n \neq m \text{ and } i, i' \in \{u, l\}\}$. Then $E(\lambda)$ is a countable set.

PROOF. Define a function g on D such that $g(z) = n$ for $z \in D_n^i(\lambda)$, $g(z) = -n$ for $z \in D_n^{i'}(\lambda)$, and $g(z) = 0$ for $z \in L(\lambda)$. If $p \in E(\lambda)$, then by Lemma 1 we have that p is an ambiguous point for the function g , and by the Bagemihl Ambiguous Point Theorem [2] we have that $E(\lambda)$ is a countable set.

We are now in a position to prove that the class \mathcal{L}_1 is a subclass of \mathcal{L} . We write $f \in \mathcal{L}_1(\mathcal{L})$ to denote that f is in the class $\mathcal{L}_1(\mathcal{L})$.

THEOREM 1. If $f \in \mathcal{L}_1$ then $f \in \mathcal{L}$.

PROOF. Suppose $f \in \mathcal{L}_1$ and let $\lambda > 0$. We need to show that $L(\lambda)$ ends at points of C . Suppose that this is not the case. Then there exists a sequence of arcs $\{\gamma_n\}$ in $L(\lambda)$ and a number $\varepsilon_0 > 0$ such that for each n we have that $\gamma_n \subset \{z: |z| > 1 - (1/n)\}$ and the diameter of γ_n exceeds ε_0 . Further, we may assume that there exists an arc Γ of C with $|\Gamma| > \varepsilon_0/2$ such that for each point $p \in \Gamma$ and each Jordan arc γ_p in $D \cup \{p\}$ with endpoint at p we have $\gamma_p \cap \gamma_n \neq \emptyset$ for all but a finite number of n . Let λ_0 be an admissible value for f such that $\lambda_0 > \lambda$. We now have two possibilities: (a) the arcs γ_n occur in infinitely many different components of $D^i(\lambda_0)$, or (b) all but a finite number of the arcs γ_n appear in a single component of $D^i(\lambda_0)$.

Suppose that (a) happens. By renumbering, if necessary, we may assume that $\gamma_n \subset D_n^i(\lambda_0)$ for each n . By (3), we have for almost all $p \in \Gamma$ that $p \in \gamma_{n_p}^i(\lambda_0)$ for some integer n_p and $i \in \{l, u\}$. Thus, for $p \in \gamma_{n_p}^i(\lambda_0)$ we have from Lemma 1 that there exists a Jordan arc $\gamma_p \subset D_{n_p}^i(\lambda_0) \cup \{p\}$ having p as an endpoint. However, for almost all $p \in \Gamma$ we have $\gamma_p \cap \gamma_n \neq \emptyset$ for all but a finite number of n . Hence no single $D_{n_p}^i(\lambda_0)$ can be found to contain such a γ_p . It follows that case (a) cannot occur.

If case (b) occurs, then there exists a single integer n_0 such that $\gamma_n \subset D_{n_0}^i(\lambda_0)$ for all but a finite number of n . Thus $\Gamma \subset \gamma_{n_0}^i(\lambda_0)$. Let φ be a conformal mapping from

D onto $D_{n_0}^i(\lambda_0)$. Then, by (1) and Theorem R, the set $\varphi^{-1}(\Gamma)$ is a set of positive measure on C . But the function $g(t) = f(\varphi(t))$ is a bounded analytic function in D and the sets $\{\varphi^{-1}(\gamma_n)\}$ are all connected subsets of $L(\lambda)$ for the function g , and hence $\limsup_{n \rightarrow \infty} \text{diam } \varphi^{-1}(\gamma_n) \geq |\varphi^{-1}(\Gamma)| > 0$, which means that g is not in the class \mathcal{L} (see [7]). It follows the case (b) cannot occur either, and so we must have $f \in \mathcal{L}$.

We can now give an answer to our question.

THEOREM 2. *If $f \in \mathcal{L}_1$ and if Γ is a subarc of C such that $A_\infty \cap \Gamma = \emptyset$, then $|F(f) \cap \Gamma| = |\Gamma|$.*

PROOF. By a theorem of Plessner (see [5, Theorem 8.2, p. 147]) almost every point of C is either a Fatou point or a Plessner point, that is, a point at which each angular cluster set is the extended complex plane. We will show that if $f \in \mathcal{L}_1$ and if p is a Plessner point which is not a member of a fixed set H , where $|H| = 0$, then $p \in A_\infty$. This will establish the theorem.

Let $f \in \mathcal{L}_1$ and let $\{\lambda_k\}$ be a sequence of admissible values for f such that $\lambda_k \rightarrow \infty$. If $H_n^i(\lambda_k) = \{p \in \gamma_n^i(\lambda_k) : \text{the tangent to } \partial(D_n^i(\lambda_k))^* \text{ at } p \text{ either does not exist or is not equal to the tangent to } C \text{ at } p\}$, then $|H_n^i(\lambda_k)| = 0$ by Theorem R. Hence

$$H' = \bigcup_k \left(\bigcup_n H_n^i(\lambda_k) \cup \bigcup_m H_m^u(\lambda_k) \right)$$

satisfies $|H'| = 0$ and, if

$$H = H' \cup \bigcup_k \left(C - \left(\bigcup_n \gamma_n^i(\lambda_k) \cup \bigcup_m \gamma_m^u(\lambda_k) \right) \right),$$

then $|H| = 0$. If $p \notin H$, we have that for each k there exists an integer n_k and $i_k \in \{l, u\}$ such that $p \in \gamma_{n_k}^{i_k}(\lambda_k)$ and the tangent to $\partial(D_{n_k}^{i_k}(\lambda_k))^*$ at p coincides with the tangent to C at p . If $i_k = l$, then each angular cluster set is bounded, so p cannot be a Plessner point. Thus, if p is a Plessner point, then $i_k = u$ for each k . In this case, $p \in \gamma_{n_k}^u(\lambda_k)$, so by Lemma 1 there exists a Jordan arc $\gamma_{p,k}$ in $D_{n_k}^u(\lambda_k) \cup \{p\}$ having p as an endpoint. By renumbering, if necessary, we may assume that $D_{n_{k+1}}^u(\lambda_{k+1}) \subset D_{n_k}^u(\lambda_k)$ for each k . Both $\gamma_{p,k}$ and $\gamma_{p,k+1}$ are the subsets of $D_{n_k}^u(\lambda_k)$, so we can cut off $\gamma_{p,k}$ at some appropriate point w_k^* and delete the portion of $\gamma_{p,k}$ from w_k^* to p , join w_k^* to a point $w_{k+1} \in \gamma_{p,k+1}$ by a Jordan arc γ_k in $D_{n_k}^u(\lambda_k)$, and delete the component of $\gamma_{p,k+1} - \{w_{k+1}\}$ which does not end at p . If we do this for each k , we get a Jordan arc γ , consisting of $\bigcup_k \gamma_k$ and appropriate portions of the arcs $\gamma_{p,k}$ between w_k and w_k^* , such that $f(z) \rightarrow \infty$ as $z \rightarrow p$ along γ . (We need to be sure that the points w_k are chosen so that the diameter of the component of $\gamma_{p,k} - \{w_k\}$ having p as an endpoint goes to 0 as $k \rightarrow \infty$, and also that $\text{diam } \gamma_k \rightarrow 0$ as $k \rightarrow \infty$. The former can be accomplished by choosing the w_k appropriately, while the latter can be accomplished by carefully deforming the line segment from w_k^* to w_{k+1} to a curve γ_k , for condition (2) guarantees that an appropriate deformation is possible.) Thus $p \in A_\infty$ and the proof is complete.

We remark that the function described in Theorem D is in the class \mathcal{L} but it violates the conclusion of Theorem 2, so this function is not in the class \mathcal{L}_1 . But $\mathcal{L}_1 \subset \mathcal{L}$ by Theorem 1, so the containment is proper.

3. Examples. We now show that there exists a function which satisfies the conclusion of Theorem 2 but is not a member of \mathcal{L}_1 .

EXAMPLE 1. *There exists a function $f \in \mathcal{L}$ such that $f \notin \mathcal{L}_1$, but for each arc Γ in C for which $\Gamma \cap A_\infty = \emptyset$ we have $|\Gamma| = |\Gamma \cap F(f)|$.*

PROOF. Let $P = \{z = x + iy: 0 \leq x < 1, y = 0\}$ and for each positive integer n let

$$H_n = \{z = x + iy: x = 2n/(2n+1), |y| \leq 1/(2n+1)\},$$

and let $H = P \cup \bigcup_{n=1}^{\infty} H_n$. For each positive integer n , let

$$K_n = \left\{z = x + iy: x = (2n-1)/(2n), 1/(4n+2) \leq |y| \leq \sqrt{4n-1}/(2n)\right\},$$

and let $K = \bigcup_{n=1}^{\infty} K_n$. By a theorem of Arakelian [1] (see also [4]), there exists a function f analytic in $\{z: |z+1| < 2\}$ such that $f(z) \rightarrow 0$ as $z \rightarrow 1$ on H and $|f(z) - n| < 1/n$ for $z \in K_n$ for each n . Let $\lambda > \text{supp}\{|f(z)|: z \in H\}$, and restrict f to the unit disc D . Then the component $D'_{n_0}(\lambda)$ of $D'(\lambda)$ for which $H \subset D'_{n_0}(\lambda)$ satisfies $(2n)/(2n+1) + i/(2n+1) \in D'_{n_0}(\lambda)$ and $(2n-1)/(2n) + i/(4n+2) \notin D'_{n_0}(\lambda)$ for n sufficiently large. It follows that $|\partial D'_{n_0}(\lambda)| = \infty$. Also, $1 \in \gamma'_{n_0}(\lambda)$ and thus no large λ is an admissible value for f , since (1) is violated for all large λ . Thus $f \notin \mathcal{L}_1$, but $\{z: |z|=1, z \neq 1\} \subset \{z: |z+1| < 2\}$, so f is analytic at each point of $C - \{1\}$, $C - \{1\} \subset F(f)$ and $A_\infty \subset \{1\}$. (It is not clear whether $A_\infty = \emptyset$ or $A_\infty = \{1\}$.) Therefore, $|\Gamma| = |\Gamma \cap F(f)|$ for each arc Γ in C .

From this example we see that Theorem 2 does not give a unique answer to the question posed in the introduction.

EXAMPLE 2. *There exists a function $f \in \mathcal{L}_1$ such that A_∞ is dense in C and $|F(f)| = 0$.*

PROOF. Let $f(z) = \prod_{k=1}^{\infty} \{1 - (z/(1 - n_k^{-1}))^{n_k}\}$ where $(n_{k+1}/n_k) \rightarrow \infty$ as $k \rightarrow \infty$. By a result of Bagemihl, Erdős and Seidel [3], we have that $A_\infty = C$ and, for each $\lambda > 0$, each of the sets $D'_n(\lambda)$ is a relatively compact subset of D with $\text{diam } D'_n(\lambda) \rightarrow 0$ as $n \rightarrow \infty$. Hence $D''(\lambda)$ consists of a single component and $\gamma''(\lambda) = C$ for each $\lambda > 0$. Thus $(D''(\lambda))^* = D$ and thus each $\lambda > 0$ is an admissible value for f , so $f \in \mathcal{L}_1$. But ∞ is the only possible Fatou value of f and hence $|F(f)| = 0$ by a theorem of Privalov (see [5, Corollary 1, p. 146]).

We note that the function in Example 2 is not a normal function but it is in \mathcal{L}_1 , while the function given in Theorem D is a normal function which is not in the class \mathcal{L}_1 . Thus, there does not exist a containment relationship between the class \mathcal{L}_1 and the class of normal analytic functions.

4. Open questions. Define the class \mathcal{L}_2 to be the collection of all functions f analytic in D such that for each arc Γ in C for which $\Gamma \cap A_\infty = \emptyset$ it is true that $|\Gamma \cap F(f)| = |\Gamma|$. In view of Theorems 1, 2 and D, Example 1 and results in [7], we must have $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}$. The following is a natural question.

Question 1. Can the class \mathcal{L}_2 be characterized in terms of level sets $L(\lambda)$ and/or the sets $D'(\lambda)$ and $D''(\lambda)$?

One possible approach to this question would be to look for an appropriate modification of condition (1) while retaining (2) and (3) in the consideration of the

admissible values. We do not have a specific candidate for a modified condition (1) at this time.

Another question is suggested by Example 2.

Question 2. Does there exist a function $f \in \mathcal{L}_1$ such that A_∞ is a countable dense subset of C and $|F(f)| = 0$?

A natural candidate to answer this question affirmatively would be the elliptic modular function, which we denote by $M(z)$. Unfortunately, $M \notin \mathcal{L}_1$. To see this, we note that for each $\lambda > 1$ it is the case that each component $D_n''(\lambda)$ satisfies the property that $\gamma_n''(\lambda)$ is a single point of C (which is a point in A_∞). Thus $D'(\lambda)$ consists of a single component, and $|\gamma'(\lambda)| = 2\pi$. Let φ be a conformal mapping from D onto $D'(\lambda)$. Then $g(z) = M(\varphi(z))$ is a bounded analytic function, so $|F(g)| = 2\pi$. If $|\partial D'(\lambda)| < \infty$, then φ can be extended continuously to the closure of D , and $|\varphi^{-1}(\gamma'(\lambda) - A_\infty)| > 0$ by Theorem R. At any point of $\gamma'(\lambda) - A_\infty$ the function M has only 0 and 1 as possible asymptotic values, and thus the function g can have only 0 and 1 as asymptotic values on the set $\varphi^{-1}(\gamma'(\lambda) - A_\infty)$. But this means that $|\varphi^{-1}(\gamma'(\lambda) - A_\infty)| = 0$ according to the theorem of Privalov, in violation of our previous assertion that $|\varphi^{-1}(\gamma'(\lambda) - A_\infty)| > 0$. It follows that $|\partial D'(\lambda)| = \infty$ for $\lambda > 1$, and thus $M \notin \mathcal{L}_1$.

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