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DECOMPOSABILITY PRESERVING CURVATURE OPERATORS WITH AN APPLICATION TO EINSTEIN MANIFOLDS

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ABSTRACT. In this paper we examine curvature operators that preserve decomposability. In particular, we prove that if at each point of an Einstein manifold M the sectional curvature operator is nonsingular and preserves decomposability, and the sectional curvature is either nonnegative or nonpositive, then M is a space of nonzero constant curvature.

1. Introduction. At each point of a Riemannian manifold its curvature tensor R induces a symmetric linear transformation R_x on Λ^2 of the tangent space. The representation of this induced "curvature operator", in the form $L \wedge L$ for some linear map L, is directly related to local embeddability in Euclidean space (see [**2, 6, 8**]).

Generally speaking, a first step has been to find conditions to insure that $R = \pm L \wedge L$. In the nonsingular case, this has been solved by Vilms in dimension greater than 4 (see [**7**]).

We prove here a similar structure theorem in dimension 4. Specifically we show (see Theorem 3.8) that a nonsingular Bianchi decomposability preserving curvature operator on $\Lambda^2 V$ with $\dim V = 4$ is of the form $\pm L \wedge L$ for some symmetric linear isomorphism $L\colon V \to V$.

The techniques used are similar to those of Vilms' in that we both rely on a theorem of Chow [**1**] on the transformations of the Grassmann variety.

It is well known that Einstein manifolds of dimension 3 are spaces of constant curvature (see [**4**]). We use our structure theorem in dimension 4 and Vilms' in dimension greater than 4 to give necessary and sufficient conditions for a (connected) Einstein manifold of dimension $\geqslant 4$ to be a space of constant curvature. The condition is that at each point x of the manifold, the induced curvature operator, R_x, is nonsingular, preserves decomposability and has sectional curvature which is nonnegative or nonpositive.

2. Preliminaries. Let V be an n-dimensional real inner product space. By $\Lambda^p(V)$, $1 \leqslant p \leqslant n$, we mean the space of p-vectors of V together with the naturally induced

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inner product from V. A 2-vector $\omega \in \Lambda^2(V)$ is said to be decomposable if $\omega = x \wedge y$ where $x, y \in V$. A map η: $\Lambda^2(V) \to \Lambda^2(V)$ preserves decomposability if $\eta(\omega)$ is decomposable for each decomposable 2-vector ω.

The following relationships between elements of $\Lambda^2(V)$ and linear subspaces of V are easily verified.

LEMMA 2.1. (1) *If* $x, y \in V$, $x \neq 0$ *and* $y \neq 0$, *then* $x \wedge y \neq 0 \Leftrightarrow$ *dimension* $\operatorname{span}\{x, y\} = 2$.

(2) *If* $x, y, u, v \in V$, *then* $x \wedge y \wedge u \wedge v = 0 \Leftrightarrow$ *dimension* $\operatorname{span}\{x, y, u, v\} < 4$.

(3) *If* $x, y, u, v \in V$, $x \wedge y \neq 0$ *and* $u \wedge v \neq 0$, *then* $x \wedge y = cu \wedge v$ *for some real number* $c \neq 0 \Leftrightarrow \operatorname{span}\{x, y\} = \operatorname{span}\{u, v\}$.

(4) *If* $x, y, u, v \in V$ *with* $x \wedge y \neq 0$ *and* $u \wedge v \neq 0$, *then* $x \wedge y \wedge u \wedge v = 0 \Leftrightarrow$ $\operatorname{span}\{x, y\}$ *and* $\operatorname{span}\{u, v\}$ *contain a common line.*

(5) *Let* $\alpha, \beta \in \Lambda^2(V)$ *be decomposable. Then* $\alpha \wedge \beta = 0 \Leftrightarrow \alpha + \beta$ *is decomposable.*

The Grassmannian G of oriented 2-planes in V will be identified with the set of all decomposable vectors of length one in $\Lambda^2(V)$.

Let $V_n = \{n\text{-dimensional subspaces of } V\}$. We say that $P, Q \in V_2$ are adjacent if they contain a common line, and that a map ϕ: $V_2 \to V_2$ preserves adjacency if whenever P and Q are adjacent then so are $\phi(P)$ and $\phi(Q)$. If, in addition, $\phi(P)$ and $\phi(Q)$ being adjacent implies that P and Q are adjacent, we say that ϕ preserves adjacency both ways. We say ϕ is induced by a linear transformation if there is a linear map L: $V \to V$ such that $\phi(P) = L(P)$ for all $P \in V_2$.

A linear transformation L: $V \to V$ induces a linear transformation $L \wedge L$: $\Lambda^2(V) \to \Lambda^2(V)$ by setting $L \wedge L(x \wedge y) = Lx \wedge Ly$ and extending linearly. It is easily checked that this induced map is well defined and that if L is symmetric then so is $L \wedge L$.

A curvature operator R is a symmetric linear operator on $\Lambda^2(V)$. Its sectional curvature is the real valued function $\sigma_R(P) = \langle RP, P\rangle$, $P \in G$. We say that R satisfies the Bianchi identity if

$$\langle Rx \wedge y, z \wedge v\rangle + \langle Ry \wedge z, x \wedge v\rangle + \langle Rz \wedge x, y \wedge v\rangle = 0 \quad \forall x, y, z, v \in V.$$

When V is oriented, has dimension 4 and has orthonormal basis $\{e_1, \ldots, e_4\}$ consistent with the given orientation, we can define the Hodge star operator $*$: $\Lambda^2(V) \to \Lambda^2(V)$ by $\langle * \alpha, \beta\rangle = \langle \alpha \wedge \beta, e_1 \wedge \cdots \wedge e_4\rangle$ where $\alpha, \beta \in \Lambda^2(V)$. It is easily checked that this definition is independent of the choice of oriented orthonormal basis for V. We note that $*$ does not satisfy the Bianchi identity.

Let dimension $V = 4$. We define $\perp$: $V_2 \to V_2$ by $\perp(P) = \{v \in V$: $\langle v, w\rangle = 0$ $\forall w \in P\}$ and $\perp'$: $V_3 \to V_1$ by $\perp'(Q) = \{v \in V$: $\langle v, w\rangle = 0\ \forall w \in Q\}$.

3. Algebraic results. In this section, for dimension $V = 4$, we give necessary and sufficient conditions for a curvature operator to be of the form $\pm L \wedge L$ where L is a nonsingular symmetric linear transformation of V onto itself.

The following theorem is a special case of a result by Wei-Liang Chow [**1**].

THEOREM 3.1. *Let V be a 4-dimensional real vector space. If ϕ: $V_2 \to V_2$ is* 1-1, *onto and preserves adjacency both ways, then either*

(1) $\exists \alpha$: $V_1 \to V_1$ *which is* 1-1, *onto and such that* $\forall l \in V_1$, $\{\phi(P)\colon P \supset l\} = \{P \in V_2\colon P \supset \alpha(l)\}$ *or*

(2) $\exists \beta$: $V_1 \to V_3$ *which is* 1-1, *onto and such that* $\forall l \in V_1$, $\{\phi(P)\colon P \supset l\} = \{P \in V_2\colon P \subset \beta(l)\}$.

Moreover, (1) *occurs* $\Leftrightarrow \exists$ *a nonsingular linear map* L: $V \to V$ *such that for all* $P \in V_2$, $\phi(P) = L(P)$.

COROLLARY 3.2. *With the assumptions and notation as in the theorem, assume V is endowed with an inner product $\langle\ ,\ \rangle$. Then $\exists$ a linear isomorphism L: $V \to V$ such that either $\phi(P) = L(P)$ $\forall P \in V_2$ or $(\perp \circ \phi)(P) = L(P)$ $\forall P \in V_2$.*

PROOF. If (1) as in the theorem occurs, then we are done. Assume (2) occurs. Then $\exists \beta$: $V_1 \to V_3$ such that $\{\phi(P)\colon P \supset l\} = \{P \in V_2\colon P \subset \beta(l)\}$. Consider $\psi = \perp \circ \phi$. It is easily checked that ψ is 1-1, onto and preserves adjacency both ways. Since (2) holds for ϕ, $\{(\perp \circ \phi)(P)\colon P \supset l\} = \{\perp(P)\colon P \subset \beta(l)\}$. But $\{\perp(P)\colon P \subset \beta(l)\} = \{P\colon P \supset (\perp' \circ \beta)(l)\}$. Thus (1) holds for ψ with $\alpha = \perp' \circ \beta$.

Given a linear isomorphism R: $\Lambda^2(V) \to \Lambda^2(V)$ which preserves decomposability, define $[R]$: $V_2 \to V_2$ by $[R]$ $(\operatorname{span}\{x, y\}) = \operatorname{span}\{u, v\}$ where $R(x \wedge y) = \mu \wedge v$. That this map is well defined follows from Lemma 2.1.

PROPOSITION 3.3. *If R: $\Lambda^2(V) \to \Lambda^2(V)$ is a linear isomorphism which preserves decomposability, then*:

(1) $[R]$ *is* 1-1.

(2) $[R]$ *preserves adjacency.*

(3) *If either* (a) *$R(\alpha)$ decomposable $\Rightarrow \alpha$ decomposable or* (b) *$\forall$ decomposable $\alpha, \beta \in \Lambda^2(V)$, $R\alpha \wedge R\beta = 0 \Rightarrow \alpha \wedge \beta = 0$ or* (c) *R is symmetric, then $[R]$ is onto and preserves adjacency both ways.*

PROOF. Part (1) follows from Lemma 2.1.

To prove (2) suppose that P and Q are adjacent. Then $P = \operatorname{span}\{m, l\}$ and $Q = \operatorname{span}\{n, l\}$ for some $m, n, l \in V$. By Lemma 2.1(4), since R preserves decomposability we need only show that $R(m \wedge l) \wedge R(n \wedge l) = 0$. But $R(m \wedge l + n \wedge l)$ is decomposable. So $R(m \wedge l + n \wedge l) \wedge R(m \wedge l + n \wedge l) = 0$. But then

$$2R(m \wedge l) \wedge R(n \wedge l) = 0$$

and (2) follows.

Now we will prove (3). That hypotheses (a) and (b) are equivalent follows from Lemma 2.1(5). In **[8]**, Vilms shows explicitly that hypothesis (c) implies hypothesis (b). Thus we need only show (3) follows under hypothesis (a). Under (a) it is easy to check that $[R]$ is onto. So, let $P = \operatorname{span}\{x, y\}$ and $Q = \operatorname{span}\{u, v\}$ be such that $[R](P)$ and $[R](Q)$ are adjacent. We must show that P and Q are adjacent.

Let l be a vector common to $[R](P)$ and $[R](Q)$. Then

$$\begin{aligned} R(x \wedge y + u \wedge v) &= R(x \wedge y) + R(u \wedge v) = m \wedge l + n \wedge l \\ &= (m + n) \wedge l \quad \text{for some } m, n \in V. \end{aligned}$$

Thus $R(x \wedge y + u \wedge v)$ is decomposable and hence, by assumption, so is

$$x \wedge y + u \wedge v.$$

So

$$0 = (x \wedge y + u \wedge v) \wedge (x \wedge y + u \wedge v) = 2(x \wedge y \wedge u \wedge v).$$

But then P and Q are adjacent by Lemma 2.1(4).

PROPOSITION 3.4 (VILMS [**8**]). *Let* $R\colon \Lambda^2(V) \to \Lambda^2(V)$ *be a linear isomorphism which preserves decomposability. Assume* $[R]$ *is induced by a linear map. Then* $R = \pm L \wedge L$ *for some linear transformation* $L\colon V \to V$.

LEMMA 3.5. *Let* V *be a* 4*-dimensional real inner product space and* $*\colon \Lambda^2(V) \to \Lambda^2(V)$ *and* $\perp\colon V_2 \to V_2$ *be defined as in* §2. *Then* $[*] = \perp$.

PROOF. Let $P \in V_2$ and $\{u_1, \ldots, u_4\}$ be an orthonormal basis for V such that $P = \operatorname{span}\{u_1, u_2\}$. Checking that $*(u_1 \wedge u_2) = cu_3 \wedge u_4$ for some real number c and applying Lemma 2.1(3) completes the proof.

THEOREM 3.6. *Let* V *be a* 4*-dimensional real inner product space and* $R\colon \Lambda^2(V) \to \Lambda^2(V)$ *a linear isomorphism. Assume* R *preserves decomposability and that* $[R]$ *is onto and preserves adjacency both ways. Then there exists a linear isomorphism* $L\colon V \to V$ *such that either* $R = \pm L \wedge L$ *or* $R = \pm * L \wedge L$.

PROOF. Corollary 3.2 applies to $[R]$ and so there exists a linear isomorphism M: $V \to V$ such that either $[R](P) = M(P)\ \forall P \in V_2$, or $(\perp \circ [R])(P) = M(P)\ \forall P \in V_2$. By Proposition 3.4 and Lemma 3.5 $R = \pm L \wedge L$ or $*R = \pm L \wedge L$ for some linear map $L\colon V \to V$. Noting that $*^2 =$ identity completes the proof.

PROPOSITION 3.7 (VILMS [7]). *Let dimension* $V \geq 3$. *Assume that* $R\colon \Lambda^2(V) \to \Lambda^2(V)$ *is a nonsingular curvature operator satisfying the Bianchi identity. If* $R = \pm L \wedge L$, *then* L *is symmetric.*

THEOREM 3.8. *Let* V *be a* 4*-dimensional real inner product space. A curvature operator* R *is*

(1) *nonsingular*,

(2) *satisfies the Bianchi identity*,

(3) *preserves decomposability*

if and only if there exists a symmetric linear isomorphism $L\colon V \to V$ *such that* $R = \pm L \wedge L$.

PROOF. By Theorem 3.6, $R = \pm L \wedge L$ or $R = \pm * L \wedge L$. We will show that $R = \pm * L \wedge L$ contradicts assumption (2). So, assume $R = \pm * L \wedge L$. Then

$$\begin{aligned}\langle * L \wedge L(x \wedge y), z \wedge v\rangle &+ \langle * L \wedge L(y \wedge z), x \wedge v\rangle \\ &+ \langle * L \wedge L(z \wedge x), y \wedge v\rangle = 0 \quad \forall x, y, z, v \in V.\end{aligned}$$

Equivalently, with δ a generator of $\Lambda^4(V)$,

$$\begin{aligned}\langle Lx \wedge Ly \wedge z \wedge v, \delta\rangle &+ \langle Ly \wedge Lz \wedge x \wedge v, \delta\rangle \\ &+ \langle Lz \wedge Lx \wedge y \wedge v, \delta\rangle = 0 \quad \forall x, y, z, v \in V.\end{aligned}$$

Setting $Lx = v$ gives $\langle Ly \wedge Lz \wedge x \wedge Lx, \delta \rangle = 0\ \forall x, y, z \in V$. Thus,

$$Ly \wedge Lz \wedge x \wedge Lx = 0 \quad \forall x, y, z \in V.$$

It follows by Lemma 2.1(2) that $\{x, Lx, Ly, Lz\}$ is a dependent set of vectors.

We next show that for each $x \in V$, $Lx = C_x x$ for some real number C_x. Fix x and assume x and Lx are independent. So, $\{x, Lx\}$ can be extended to a basis $\{x, Lx, u, v\}$ for V. Since L is nonsingular there exist $y, z \in V$ such that $Ly = u$ and $Lz = V$. But then $\{x, Lx, u, v\} = \{x, Lx, Ly, Lz\}$ is a dependent set of vectors, which is a contradiction.

So, indeed, $\forall x \in V$, $Lx = C_x x$ for some real number C_x.

We now show C_x does not depend on x. Assume x and y are independent. By the linearity of L, $C_x x + C_y y = C_{x+y} x + C_{x+y} y$. Since x and y are independent we have that $C_x = C_{x+y} = C_y$. But then $R = \pm k^*$ for some real numbers $k \neq 0$. This contradicts assumption (2). That the converse holds is easily checked.

REMARK. The above result for $\dim V > 4$ appears in [7].

4. Geometric applications. In this section we give necessary and sufficient conditions for an n-dimensional Einstein manifold, $n \geqslant 4$, to be a space of constant curvature k, $k \neq 0$.

Given a curvature operator $R\colon \Lambda^2(V) \to \Lambda^2(V)$, its Ricci contraction $r(R)$ is the symmetric linear transformation on V given by

$$\langle r(R)(v), w \rangle = \sum_i \langle R(v \wedge e_i), w \wedge e_i \rangle$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V. It is easily checked that this definition is basis free.

A Riemannian manifold is called an Einstein manifold if its Ricci tensor is a constant multiple of the metric tensor. At a point this condition implies that $r(R) = kI$ for some real number k.

THEOREM 4.1. *Let M be an n-dimensional (connected) Einstein manifold, $n \geqslant 4$, and let R be the induced curvature operator at some point of M. Assume R is nonsingular and preserves decomposability. Then*

(a) $\sigma_R \geqslant 0 \Rightarrow R = kI \wedge I$ *with* $k > 0$,

(b) $\sigma_R \leqslant 0 \Rightarrow R = kI \wedge I$ *with* $k < 0$.

PROOF. By Theorem 3.8, $R = \pm L \wedge L$ where L is symmetric and nonsingular. So there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for V such that $Le_i = a_i e_i$, $a_i \neq 0$. Then $\{e_i \wedge e_j\colon i < j\}$ is an orthonormal basis for $\Lambda^2(V)$ such that $Re_i \wedge e_j = a_i a_j e_i \wedge e_j\ \forall i, j$ or $Re_i \wedge e_j = -a_i a_j e_i \wedge e_j\ \forall i, j$. It suffices to show that $a_k = a_l\ \forall k, l$. Since $r(R)$ is a multiple of the identity, it follows that, for some constant, c, $\langle r(R)e_j, e_j \rangle = c$ for each j. Equivalently, $c = \Sigma_{i \neq j} a_j a_i$ for each j. Now, let $k \neq l$. Then $\Sigma_{i \neq k} a_k a_i = \Sigma_{i \neq l} a_l a_i$. Set $S(k, l) = a_1 + \cdots + \hat{a}_k + \cdots + \hat{a}_l + \cdots + a_n$. Thus $a_k S(k, l) = a_l S(k, l)$.

We will be done if we show $S(k, l) \neq 0$. This will certainly be true if we show that $a_1, \ldots, a_n$ have the same sign.

First assume $\sigma_R \geqslant 0$. We know that $R = \pm L \wedge L$. We claim $R = +L \wedge L$, for if $R = -L \wedge L$ then $0 \leqslant \sigma_R(e_i \wedge e_j) = -a_i a_j$ $\forall i, j$. Since L is nonsingular, we have $a_i \neq 0$ $\forall i$, so that, in fact, $0 < -a_i a_j$ $\forall i, j$. But this is impossible as $n \geqslant 3$. So $R = +L \wedge L$. Since $\sigma_R \geqslant 0$ and L is nonsingular, we now have $0 < a_i a_j$ $\forall i, j$ and thus, when $\sigma_R \geqslant 0$, $a_1, \ldots, a_n$ are of the same sign.

Now assume $\sigma_R \leqslant 0$. An argument similar to the one above gives that $R = -L \wedge L$, which together with the nonsingular of L implies that $0 > -a_i a_j$ $\forall i, j$. And so, when $\sigma_R \leqslant 0$, $a_1, \ldots, a_n$ are of the same sign.

Thus, in either case, $a_1, \ldots, a_n$ have the same sign, so the theorem follows.

COROLLARY 4.2. *Let M be an n-dimensional Einstein manifold with $n \geqslant 4$. Then the following two statements are equivalent.*

(A) *M is a space of nonzero constant curvature.*

(B) *At each point the curvature operator is nonsingular, preserves decomposability and has sectional curvature which is either nonnegative or nonpositive.*

PROOF. That (A) implies (B) is clear. We now show that (B) implies (A).

For each $x \in M$, let R_x denote the sectional curvature at x. Then, by Theorem 4.1, $R_x = k_x I \wedge I$ for some nonzero constant k_x. So $\sigma_{R_x}(P)$, for $P \in G$, depends only on x and not on P. So by a theorem of Schur [**3**, **5**], M is a space of nonzero constant curvature.

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