DECOMPOSABILITY PRESERVING CURVATURE OPERATORS WITH AN APPLICATION TO EINSTEIN MANIFOLDS

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Abstract. In this paper we examine curvature operators that preserve decomposability. In particular, we prove that if at each point of an Einstein manifold \( M \) the sectional curvature operator is nonsingular and preserves decomposability, and the sectional curvature is either nonnegative or nonpositive, then \( M \) is a space of nonzero constant curvature.

1. Introduction. At each point of a Riemannian manifold its curvature tensor \( R \) induces a symmetric linear transformation \( R_x \) on \( \Lambda^2 \) of the tangent space. The representation of this induced “curvature operator”, in the form \( L \wedge L \) for some linear map \( L \), is directly related to local embeddability in Euclidean space (see [2, 6, 8]).

Generally speaking, a first step has been to find conditions to insure that \( R = \pm L \wedge L \). In the nonsingular case, this has been solved by Vilms in dimension greater than 4 (see [7]).

We prove here a similar structure theorem in dimension 4. Specifically we show (see Theorem 3.8) that a nonsingular Bianchi decomposability preserving curvature operator on \( \Lambda^2 V \) with \( \dim V = 4 \) is of the form \( \pm L \wedge L \) for some symmetric linear isomorphism \( L: V \to V \).

The techniques used are similar to those of Vilms’ in that we both rely on a theorem of Chow [1] on the transformations of the Grassmann variety.

It is well known that Einstein manifolds of dimension 3 are spaces of constant curvature (see [4]). We use our structure theorem in dimension 4 and Vilms’ in dimension greater than 4 to give necessary and sufficient conditions for a (connected) Einstein manifold of dimension \( \geq 4 \) to be a space of constant curvature. The condition is that at each point \( x \) of the manifold, the induced curvature operator, \( R_x \), is nonsingular, preserves decomposability and has sectional curvature which is nonnegative or nonpositive.

2. Preliminaries. Let \( V \) be an \( n \)-dimensional real inner product space. By \( \Lambda^p(V) \), \( 1 \leq p \leq n \), we mean the space of \( p \)-vectors of \( V \) together with the naturally induced...
inner product from $V$. A 2-vector $\omega \in \Lambda^2(V)$ is said to be decomposable if $\omega = x \wedge y$ where $x, y \in V$. A map $\eta: \Lambda^2(V) \to \Lambda^2(V)$ preserves decomposability if $\eta(\omega)$ is decomposable for each decomposable 2-vector $\omega$.

The following relationships between elements of $\Lambda^2(V)$ and linear subspaces of $V$ are easily verified.

**Lemma 2.1.** (1) If $x, y \in V$, $x \neq 0$ and $y \neq 0$, then $x \wedge y \neq 0 \iff$ dimension $\text{span}\{x, y\} = 2$.
(2) If $x, y, u, v \in V$, then $x \wedge y \wedge u \wedge v = 0 \iff$ dimension $\text{span}\{x, y, u, v\} < 4$.
(3) If $x, y, u, v \in V$, $x \wedge y \neq 0$ and $u \wedge v \neq 0$, then $x \wedge y = cu \wedge v$ for some real number $c \neq 0 \iff \text{span}\{x, y\} = \text{span}\{u, v\}$.
(4) If $x, y, u, v \in V$ with $x \wedge y \neq 0$ and $u \wedge v \neq 0$, then $x \wedge y \wedge u \wedge v = 0 \iff \text{span}\{x, y\}$ and $\text{span}\{u, v\}$ contain a common line.
(5) Let $\alpha, \beta \in \Lambda^2(V)$ be decomposable. Then $\alpha \wedge \beta = 0 \iff \alpha + \beta$ is decomposable.

The Grassmannian $G$ of oriented 2-planes in $V$ will be identified with the set of all decomposable vectors of length one in $\Lambda^2(V)$.

Let $V_n = \{n\text{-dimensional subspaces of } V\}$. We say that $P, Q \in V_2$ are adjacent if they contain a common line, and that a map $\phi: V_2 \to V_2$ preserves adjacency if whenever $P$ and $Q$ are adjacent then so are $\phi(P)$ and $\phi(Q)$. If, in addition, $\phi(Q)$ being adjacent implies that $P$ and $Q$ are adjacent, we say that $\phi$ preserves adjacency both ways. We say $\phi$ is induced by a linear transformation if there is a linear map $L: V \to F$ such that $\phi(P) = L(P)$ for all $P \in V_2$.

A linear transformation $L: V \to V$ induces a linear transformation $L \wedge L: \Lambda^2(V) \to \Lambda^2(V)$ by setting $L \wedge L(x \wedge y) = Lx \wedge Ly$ and extending linearly. It is easily checked that this induced map is well defined and that if $L$ is symmetric then so is $L \wedge L$.

A curvature operator $R$ is a symmetric linear operator on $\Lambda^2(V)$. Its sectional curvature is the real valued function $\sigma_R(P) = \langle RP, P \rangle$, $P \in G$. We say that $R$ satisfies the Bianchi identity if

$$\langle Rx \wedge y, z \wedge v \rangle + \langle Ry \wedge z, x \wedge v \rangle + \langle Rz \wedge x, y \wedge v \rangle = 0 \ \forall x, y, z, v \in V.$$ 

When $V$ is oriented, has dimension 4 and has orthonormal basis $\{e_1, \ldots, e_4\}$ consistent with the given orientation, we can define the Hodge star operator $\ast: \Lambda^2(V) \to \Lambda^2(V)$ by $\langle \ast \alpha, \beta \rangle = \langle \alpha \wedge \beta, e_1 \wedge \cdots \wedge e_4 \rangle$ where $\alpha, \beta \in \Lambda^2(V)$. It is easily checked that this definition is independent of the choice of oriented orthonormal basis for $V$. We note that $\ast$ does not satisfy the Bianchi identity.

Let dimension $V = 4$. We define $\perp: V_2 \to V_2$ by $\perp(P) = \{v \in V: \langle v, w \rangle = 0 \ \forall w \in P\}$ and $\perp': V_3 \to V_1$ by $\perp'(Q) = \{v \in V: \langle v, w \rangle = 0 \ \forall w \in Q\}$.

**3. Algebraic results.** In this section, for dimension $V = 4$, we give necessary and sufficient conditions for a curvature operator to be of the form $\pm L \wedge L$ where $L$ is a nonsingular symmetric linear transformation of $V$ onto itself.

The following theorem is a special case of a result by Wei-Liang Chow [1].
Theorem 3.1. Let $V$ be a 4-dimensional real vector space. If $\phi: V_2 \to V_2$ is 1-1, onto and preserves adjacency both ways, then either

(1) $\exists \alpha: V_1 \to V_1$ which is 1-1, onto and such that $\forall l \in V_1$, $\{\phi(P): P \supset l\} = \{P \in V_2: P \supset \alpha(l)\}$ or

(2) $\exists \beta: V_1 \to V_1$ which is 1-1, onto and such that $\forall l \in V_1$, $\{\phi(P): P \supset l\} = \{P \in V_2: P \subset \beta(l)\}$.

Moreover, (1) occurs ⇔ $\exists$ a nonsingular linear map $L: V \to V$ such that for all $P \in V_2$, $\phi(P) = L(P)$.

Corollary 3.2. With the assumptions and notation as in the theorem, assume $V$ is endowed with an inner product $(\cdot, \cdot)$. Then $\exists$ a linear isomorphism $L: V \to V$ such that either $\phi(P) = L(P) \forall P \in V_2$ or $(\perp \circ \phi)(P) = L(P) \forall P \in V_2$.

Proof. If (1) as in the theorem occurs, then we are done. Assume (2) occurs. Then $\exists \beta: V_1 \to V_1$ such that $\{\phi(P): P \supset l\} = \{P \subset \beta(l)\}$. Consider $\psi = \perp \circ \phi$. It is easily checked that $\psi$ is 1-1, onto and preserves adjacency both ways. Since (2) holds for $\phi$, $\{(\perp \circ \phi)(P): P \supset l\} = \{(P): P \subset \beta(l)\}$. But $\{(P): P \subset \beta(l)\} = \{P: P \supset (\perp \circ \phi)(l)\}$. Thus (1) holds for $\psi$ with $\alpha = \perp \circ \beta$.

Given a linear isomorphism $R: \Lambda^2(V) \to \Lambda^2(V)$ which preserves decomposability, define $[R]: V_2 \to V_2$ by $[R](\text{span}\{x, y\}) = \text{span}\{u, v\}$ where $R(x \wedge y) = u \wedge v$. That this map is well defined follows from Lemma 2.1.

Proposition 3.3. If $R: \Lambda^2(V) \to \Lambda^2(V)$ is a linear isomorphism which preserves decomposability, then:

1. $[R]$ is 1-1.

2. $[R]$ preserves adjacency.

3. If either (a) $R(\alpha)$ decomposable $\Rightarrow \alpha$ decomposable or (b) $\forall$ decomposable $\alpha, \beta \in \Lambda^2(V)$, $R \alpha \wedge R \beta = 0 = \alpha \wedge \beta = 0$ or (c) $R$ is symmetric, then $[R]$ is onto and preserves adjacency both ways.

Proof. Part (1) follows from Lemma 2.1.

To prove (2) suppose that $P$ and $Q$ are adjacent. Then $P = \text{span}\{m, l\}$ and $Q = \text{span}\{n, l\}$ for some $m, n, l \in V$. By Lemma 2.1(4), since $R$ preserves decomposability we need only show that $R(m \wedge l) \wedge R(n \wedge l) = 0$. But $R(m \wedge l + n \wedge l)$ is decomposable. So $R(m \wedge l + n \wedge l) \wedge R(m \wedge l + n \wedge l) = 0$. But then

$$2R(m \wedge l) \wedge R(n \wedge l) = 0$$

and (2) follows.

Now we will prove (3). That hypotheses (a) and (b) are equivalent follows from Lemma 2.1(5). In [8], Vilms shows explicitly that hypothesis (c) implies hypothesis (b). Thus we need only show (3) follows under hypothesis (a). Under (a) it is easy to check that $[R]$ is onto. So, let $P = \text{span}\{x, y\}$ and $Q = \text{span}\{u, v\}$ be such that $[R](P)$ and $[R](Q)$ are adjacent. We must show that $P$ and $Q$ are adjacent.

Let $l$ be a vector common to $[R](P)$ and $[R](Q)$. Then

$$R(x \wedge y + u \wedge v) = R(x \wedge y) + R(u \wedge v) = m \wedge l + n \wedge l$$

$$= (m + n) \wedge l \text{ for some } m, n \in V.$$
Thus $R(x \wedge y + u \wedge v)$ is decomposable and hence, by assumption, so is $x \wedge y + u \wedge v$.

So

$$0 = (x \wedge y + u \wedge v) \wedge (x \wedge y + u \wedge v) = 2(x \wedge y \wedge u \wedge v).$$

But then $P$ and $Q$ are adjacent by Lemma 2.1(4).

**Proposition 3.4 (Vilms [8]).** Let $R: \Lambda^2(V) \to \Lambda^2(V)$ be a linear isomorphism which preserves decomposability. Assume $[R]$ is induced by a linear map. Then $R = \pm L \wedge L$ for some linear transformation $L: V \to V$.

**Lemma 3.5.** Let $V$ be a 4-dimensional real inner product space and $*: \Lambda^3(V) \to \Lambda^3(V)$ and $\perp: V_2 \to V_2$ be defined as in §2. Then $[\ast] = \perp$.

**Proof.** Let $P \in V_2$ and $\{u_1, \ldots, u_4\}$ be an orthonormal basis for $V$ such that $P = \text{span}\{u_1, u_2\}$. Checking that $\ast (u_1 \wedge u_2) = cu_3 \wedge u_4$ for some real number $c$ and applying Lemma 2.1(3) completes the proof.

**Theorem 3.6.** Let $V$ be a 4-dimensional real inner product space and $R: \Lambda^2(V) \to \Lambda^2(V)$ a linear isomorphism. Assume $R$ preserves decomposability and that $[R]$ is onto and preserves adjacency both ways. Then there exists a linear isomorphism $L: V \to V$ such that either $R = \pm L \wedge L$ or $R = \pm \ast L \wedge L$.

**Proof.** Corollary 3.2 applies to $[R]$ and so there exists a linear isomorphism $M: V \to V$ such that either $[R](P) = M(P) \quad \forall P \in V_2$, or $(\perp \circ [R])(P) = M(P) \quad \forall P \in V_2$. By Proposition 3.4 and Lemma 3.5 $R = \pm L \wedge L$ or $\ast R = \pm L \wedge L$ for some linear map $L: V \to V$. Noting that $\ast^2 = \text{identity}$ completes the proof.

**Proposition 3.7 (Vilms [7]).** Let dimension $V \geq 3$. Assume that $R: \Lambda^2(V) \to \Lambda^2(V)$ is a nonsingular curvature operator satisfying the Bianchi identity. If $R = \pm L \wedge L$, then $L$ is symmetric.

**Theorem 3.8.** Let $V$ be a 4-dimensional real inner product space. A curvature operator $R$ is

1. nonsingular,
2. satisfies the Bianchi identity,
3. preserves decomposability

if and only if there exists a symmetric linear isomorphism $L: V \to V$ such that $R = \pm L \wedge L$.

**Proof.** By Theorem 3.6, $R = \pm L \wedge L$ or $R = \pm \ast L \wedge L$. We will show that $R = \pm \ast L \wedge L$ contradicts assumption (2). So, assume $R = \pm \ast L \wedge L$. Then

$$(\ast L \wedge L(x \wedge y), z \wedge v) + (\ast L \wedge L(y \wedge z), x \wedge v)$$

$$+ (\ast L \wedge L(z \wedge x), y \wedge v) = 0 \quad \forall x, y, z, v \in V.$$

Equivalently, with $\delta$ a generator of $\Lambda^4(V)$,

$$(Lx \wedge Ly \wedge z \wedge v, \delta) + (Ly \wedge Lz \wedge x \wedge v, \delta)$$

$$+ (Lz \wedge Lx \wedge y \wedge v, \delta) = 0 \quad \forall x, y, z, v \in V.$$
Setting $Lx = v$ gives $\langle Ly \wedge Lz \wedge x \wedge Lx, \delta \rangle = 0 \ \forall x, y, z \in V$. Thus, 
$$Ly \wedge Lz \wedge x \wedge Lx = 0 \ \forall x, y, z \in V.$$ 
It follows by Lemma 2.1(2) that $\{x, Lx, Ly, Lz\}$ is a dependent set of vectors.

We next show that for each $x \in V$, $Lx = C_x x$ for some real number $C_x$. Fix $x$ and assume $x$ and $Lx$ are independent. So, $\{x, Lx\}$ can be extended to a basis $\{x, Lx, u, v\}$ for $V$. Since $L$ is nonsingular there exist $y, z \in V$ such that $Ly = u$ and $Lz = V$. But then $\{x, Lx, u, v\} = \{x, Lx, Ly, Lz\}$ is a dependent set of vectors, which is a contradiction.

So, indeed, $\forall x \in V, Lx = C_x x$ for some real number $C_x$.

We now show $C_x$ does not depend on $x$. Assume $x$ and $y$ are independent. By the linearity of $L$, $C_x x + C_y y = C_{x+y} x + C_{x+y} y$. Since $x$ and $y$ are independent we have that $C_x = C_{x+y} = C_y$. But then $R = \pm k^* \ \forall k \neq 0$. This contradicts assumption (2). That the converse holds is easily checked.

Remark. The above result for $\dim V > 4$ appears in [7].

4. Geometric applications. In this section we give necessary and sufficient conditions for an $n$-dimensional Einstein manifold, $n \geq 4$, to be a space of constant curvature $k, k \neq 0$.

Given a curvature operator $R: \Lambda^2(K) \to \Lambda^2(K)$, its Ricci contraction $r(R)$ is the symmetric linear transformation on $V$ given by

$$\langle r(R)(v), w \rangle = \sum_i \langle R(v \wedge e_i), w \wedge e_i \rangle$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $V$. It is easily checked that this definition is basis free.

A Riemannian manifold is called an Einstein manifold if its Ricci tensor is a constant multiple of the metric tensor. At a point this condition implies that $r(R) = kI$ for some real number $k$.

Theorem 4.1. Let $M$ be an $n$-dimensional (connected) Einstein manifold, $n \geq 4$, and let $R$ be the induced curvature operator at some point of $M$. Assume $R$ is nonsingular and preserves decomposability. Then

(a) $\sigma_R \geq 0 \Rightarrow R = kI \wedge I$ with $k > 0$,
(b) $\sigma_R \leq 0 \Rightarrow R = kI \wedge I$ with $k < 0$.

Proof. By Theorem 3.8, $R = \pm L \wedge I$ where $L$ is symmetric and nonsingular. So there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for $V$ such that $Le_i = a_i e_i, a_i \neq 0$. Then $\{e_i \wedge e_j: i < j\}$ is an orthonormal basis for $\Lambda^2(V)$ such that $Re_i \wedge e_j = a_i a_j e_i \wedge e_j \ \forall i, j$ or $Re_j \wedge e_i = -a_i a_j e_i \wedge e_j \ \forall i, j$. It suffices to show that $a_k = a_l \ \forall k, l$. Since $r(R)$ is a multiple of the identity, it follows that, for some constant, $c$, $\langle r(R)e_j, e_j \rangle = c \ \forall j$. Equivalently, $c = \sum_i a_i a_i$ for each $j$. Now, let $k \neq l$. Then $\sum_i a_k a_i = \sum_i a_l a_i$. Set $S(k, l) = a_l + \cdots + a_k + \cdots + a_i + \cdots + a_n$. Thus $a_k S(k, l) = a_l S(k, l)$.

We will be done if we show $S(k, l) \neq 0$. This will certainly be true if we show that $a_1, \ldots, a_n$ have the same sign.
First assume $\sigma_R \geq 0$. We know that $R = \pm L \wedge L$. We claim $R = +L \wedge L$, for if $R = -L \wedge L$ then $0 \leq \sigma_R(e_i \wedge e_j) = -a_i a_j \forall i, j$. Since $L$ is nonsingular, we have $a_i \neq 0 \forall i$, so that, in fact, $0 < -a_i a_j \forall i, j$. But this is impossible as $n \geq 3$. So $R = +L \wedge L$. Since $\sigma_R \geq 0$ and $L$ is nonsingular, we now have $0 < a_i a_j \forall i, j$ and thus, when $\sigma_R \geq 0$, $a_1, \ldots, a_n$ are of the same sign.

Now assume $\sigma_R \leq 0$. An argument similar to the one above gives that $R = -L \wedge L$, which together with the nonsingular of $L$ implies that $0 > -a_i a_j \forall i, j$. And so, when $\sigma_R \leq 0$, $a_1, \ldots, a_n$ are of the same sign.

Thus, in either case, $a_1, \ldots, a_n$ have the same sign, so the theorem follows.

**COROLLARY 4.2.** Let $M$ be an $n$-dimensional Einstein manifold with $n \geq 4$. Then the following two statements are equivalent.

(A) $M$ is a space of nonzero constant curvature.

(B) At each point the curvature operator is nonsingular, preserves decomposability and has sectional curvature which is either nonnegative or nonpositive.

**PROOF.** That (A) implies (B) is clear. We now show that (B) implies (A).

For each $x \in M$, let $R_x$ denote the sectional curvature at $x$. Then, by Theorem 4.1, $R_x = k_x I \wedge I$ for some nonzero constant $k_x$. So $\sigma_R(P)$, for $P \in G$, depends only on $x$ and not on $P$. So by a theorem of Schur [3, 5], $M$ is a space of nonzero constant curvature.

**References**


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