PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 2, February 1984

NOTE ON EPIMORPHISMS AND MONOMORPHISMS IN HOMOTOPY THEORY

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ABSTRACT. We study epimorphisms $e: X \to X$ and monomorphisms $m: X \to X$ in the pointed homotopy category of path-connected CW-spaces. Our principal theorems allow us to infer that under suitable finiteness and fundamental group assumptions, such are in fact homotopy equivalences.

1. Recall that a group G is said to be *Hopfian* if every epimorphism $\varepsilon: G \twoheadrightarrow G$ is an automorphism. The concept was introduced by Heinz Hopf [**H**] in his study of the classification of surface mappings and has proved of interest in group theory; for example, Graham Higman presented the first example of a finitely presented non-Hopfian group. Recently certain properties of pseudo-identities of locally nilpotent groups were observed [**HR**] to be related to the basic yet elementary fact that finitely generated nilpotent groups are Hopfian.

Now there is a theory of pseudo-identities in the (pointed) homotopy category $\mathcal{H}$ of path-connected CW-spaces [**CHR**] and this motivates our study here of Hopfian objects of $\mathcal{H}$; this notion plainly makes sense in any category since epimorphisms are categorically defined. It will turn out that nilpotent spaces of finite type are, in fact, Hopfian objects of $\mathcal{H}$.

It is natural to consider the dual notion of co-Hopfian object, based on the study of self-monomorphisms. In the category $\mathcal{G}$ of groups, the most obvious co-Hopfian objects are the finite groups (these are, of course, also Hopfian), but there are other examples. For example, any nilpotent group whose p-torsion subgroup is finite for each p and such that the quotient by the torsion subgroup is a finite direct product of copies of the rationals is co-Hopfian (again, such groups are also Hopfian). It turns out that we can characterize self-monomorphisms in $\mathcal{H}$, provided that we confine attention to spaces whose higher homotopy groups are finitely generated. We deduce that if such spaces have co-Hopfian fundamental groups they are co-Hopfian objects of $\mathcal{H}$.

The adjunction equivalence

$$[X, K(G,1)] \cong \operatorname{Hom}(\pi_1 X, G), \tag{1.1}$$

Received by the editors January 24, 1983 and, in revised form, May 12, 1983.
1980 *Mathematics Subject Classification*. Primary 55D10, 55D30.
Key words and phrases. Epimorphism, monomorphism, Hopfian object, co-Hopfian object.

where [,] denotes the set of (pointed) homotopy classes of maps, leads to the immediate inference

PROPOSITION 1. *If* $e: X \to Y$ *is an epimorphism in* $\mathcal{H}$, *then* $e_*: \pi_1 X \to \pi_1 Y$ *is an epimorphism in* $\mathcal{G}$.

COROLLARY 2. *If* G *is a Hopfian group, then* $K(G, 1)$ *is a Hopfian object of* $\mathcal{H}$.

The converse of this assertion may well fail. It is easy to construct a map f: $K(G_1, 1) \to K(G_2, 1)$ which is not an epimorphism although $f_*: G_1 \twoheadrightarrow G_2$. Thus, if G_2 is a group having a non-0 cohomology group $H^i(G_2; \Gamma)$ (constant coefficients) $i \geq 2$, and if G_1 is a free group mapping onto G_2 (e.g., $\mathbf{Z} \twoheadrightarrow \mathbf{Z}/2$), then f is not an epimorphism in $\mathcal{H}$. From this example we readily construct a group G, namely $(\oplus_{i=-\infty}^{0} G_1) \oplus (\oplus_{j=1}^{\infty} G_2)$, and a "shift" map f: $K(G, 1) \to K(G, 1)$ which is not an epimorphism but such that $f_*: G \twoheadrightarrow G$. The existence of such maps f is certainly an obstacle to any simple-minded proof of a converse to Corollary 2.

Dual to Proposition 1 is

PROPOSITION 1*. *If* $\mu: G \to H$ *is a monomorphism in* $\mathcal{G}$ *then* $K(\mu, 1)$: $K(G, 1) \to K(H, 1)$ *is a monomorphism in* $\mathcal{H}$.

COROLLARY 2*. *If* $K(G, 1)$ *is a co-Hopfian object of* $\mathcal{H}$, *then* $\mathcal{G}$ *is a co-Hopfian group.*

However, the converse of Corollary 2* is true since the converse of Proposition 1* is true.

2. Epimorphisms in $\mathcal{H}$. We say that $X \in \mathcal{H}$ is *of homologically finite type* (*hft*) if $H_i X$ is finitely generated for $i \geq 1$.

THEOREM 3. *If* X *is hft and* $e: X \to X$ *is an epimorphism in* $\mathcal{H}$, *then* $e_*: H_i X \cong H_i X$, $i \geq 0$.

PROOF. For any (constant) coefficient group G, we have $e^*: H^i(X; G) \rightarrowtail H^i(X; G)$, $i \geq 0$. If we take $G = \mathbf{Z}/p$ and pass to homology, we infer that e_*: $H_i(X; \mathbf{Z}/p) \twoheadrightarrow H_i(X; \mathbf{Z}/p)$, $i \geq 0$. But $H_i(X; \mathbf{Z}/p)$ is finite since X is hft, so that $e_*: H_i(X; \mathbf{Z}/p) \cong H_i(X; \mathbf{Z}/p)$, $i \geq 0$. We now appeal to [**HMR**, Theorem II.1.14], again exploiting the fact that X is hft, to infer that $e_*: H_i X \cong H_i X$, $i \geq 0$.

COROLLARY 4. *If, in addition,* X *is nilpotent, then* $e: X \to X$ *is a homotopy equivalence.*

PROOF. We know that a homology equivalence of nilpotent spaces is a homotopy equivalence.

Theorem 3 admits the following generalization.

THEOREM 5. *Let* $f: X \to Y$, $g: Y \to X$ *be epimorphisms in* $\mathcal{H}$ *and let* X *be hft. Then* Y *is hft and* f *and* g *are homology equivalences.*

PROOF. By Theorem 3 we know that $g_* f_*$: $H_i X \cong H_i X$, $i \geq 0$. Thus $f^* g^*$: $H^i(X; G) \cong H^i(X; G)$ for any (constant) coefficient group G. But f^*: $H^i(Y; G) \rightarrowtail H^i(X; G)$, so that f^*: $H^i(Y; G) \cong H^i(X; G)$; and, of course, g^*: $H^i(X; G) \cong H^i(Y; G)$. Let Z be the mapping cone of f. Then $H^i(Z; G) = 0$, $i \geq 1$, for any (constant) coefficient group G. From this it is easy to deduce that $H_i Z = 0$, $i \geq 1$, so that f_*: $H_i X \cong H_i Y$, $i \geq 0$. Similarly, g_*: $H_i Y \cong H_i X$, $i \geq 0$.

COROLLARY 6. *If, in addition, X and Y are nilpotent, then f and g are homotopy equivalences.*

3. Monomorphisms in $\mathcal{H}$. Let p: $\tilde{X} \to X$ be a covering map. Then certainly p is a monomorphism in $\mathcal{H}$. In particular, if there is a map m: $X \to X$ inducing a monomorphism μ: $\pi \rightarrowtail \pi$ where $\pi = \pi_1 X$ and if p: $\tilde{X} \to X$ is the covering map with $\pi_1 \tilde{X} = \mu\pi$, then m factors as ph, where h: $X \to \tilde{X}$. Moreover, h is a homotopy equivalence provided that m induces an automorphism of higher homotopy groups; and, in that case, m is a monomorphism.

We now prove a converse of this, provided we impose a restriction on X. We say that $X \in \mathcal{H}$ is of homotopically finite type (Hft) if $\pi_i X$ is finitely generated for $i \geq 2$.

THEOREM 7. *If X is Hft and m: $X \to X$ is a monomorphism in $\mathcal{H}$, then $m = ph$, where h is a homotopy equivalence h: $X \to \tilde{X}$ and p: $\tilde{X} \to X$ is a covering map with $\pi_1 \tilde{X} = m_* \pi_1 X$.*

PROOF. We invoke the theory of homotopy groups with coefficients. For $n \geq 3$, we define[1]

$$\pi_n(X; G) = [M(G, n), X], \tag{3.1}$$

where $M(G, k)$ is the co-Moore space, being a 1-connected space with single nonvanishing integral cohomology group G in dimension k.

In fact, we will only need (3.1) with $G = \mathbf{Z}/p$. It is then even possible to define the pointed set

$$\pi_2(X; \mathbf{Z}/p) = [S^1 \cup_p e^2, X].$$

Recall that the cofibration sequence

$$S^1 \xrightarrow{p} S^1 \to S^1 \cup_p e^2$$

gives rise to a long exact sequence

$$\cdots \to \pi_n X \xrightarrow{p} \pi_n X \to \pi_n(X; \mathbf{Z}/p) \to \pi_{n-1} X \xrightarrow{p} \pi_{n-1} X \to \cdots \tag{3.2}$$
$$\to \pi_2(X; \mathbf{Z}/p) \to \pi_1 X \xrightarrow{p} \pi_1 X$$

[1] We depart here from the Eckmann-Hilton definition; we will only be concerned with coefficient groups G which are finitely generated.

and hence to a *universal coefficient sequence*

$$(3.3) \qquad \pi_n X \otimes \mathbf{Z}/p \rightarrowtail \pi_n(X; \mathbf{Z}/p) \twoheadrightarrow \mathrm{Tor}(\pi_{n-1}X; \mathbf{Z}/p), \qquad n \geqslant 3.$$

For $n = 2$, we still have the embedding

$$(3.4) \qquad \pi_2 X \otimes \mathbf{Z}/p \rightarrowtail \pi_2(X; \mathbf{Z}/p).$$

We propose to show that m_*: $\pi_n X \cong \pi_n X$, $n \geqslant 2$. We mimic the proof of [**HMR**, Theorem II.1.14] and show that m_{*1}: $\pi_n X \otimes \mathbf{Z}/p \cong \pi_n X \otimes \mathbf{Z}/p$ and m_{*2}: $\mathrm{Tor}(\pi_n X, \mathbf{Z}/p) \cong \mathrm{Tor}(\pi_n X, \mathbf{Z}/p)$ for all p and $n \geqslant 2$, whence, since X is Hft, the conclusion follows. We first show that m_{*1} is an automorphism. Now since m is a monomorphism, we have m_*: $\pi_n(X; \mathbf{Z}/p) \rightarrowtail \pi_n(X; \mathbf{Z}/p)$, $n \geqslant 2$. Thus, by (3.3) or (3.4), M_{*1} is a monomorphism. But, since X is Hft, $\pi_n X \otimes \mathbf{Z}/p$ is a finite group, so m_{*1} is an automorphism. We now prove that m_{*2} is an automorphism. For we have, with $n \geqslant 3$,

$$\begin{array}{ccccc}
\pi_n X \otimes \mathbf{Z}/p & \rightarrowtail & \pi_n(X; \mathbf{Z}/p) & \twoheadrightarrow & \mathrm{Tor}(\pi_{n-1}X, \mathbf{Z}/p) \\
\downarrow m_{*1} & & \downarrow m_* & & \downarrow m_{*2} \\
\pi_n X \otimes \mathbf{Z}/p & \rightarrowtail & \pi_n(X; \mathbf{Z}/p) & \twoheadrightarrow & \mathrm{Tor}(\pi_{n-1}X, \mathbf{Z}/p)
\end{array}$$

from which it follows that m_{*2} is a monomorphism. But again since X is Hft, m_{*2} is an automorphism. We have therefore established that m_*: $\pi_n X \cong \pi_n X$, $n \geqslant 2$, and, of course, m_*: $\pi_1 X \rightarrowtail \pi_1 X$. The conclusion of the theorem therefore follows.[2]

COROLLARY 8. *If, in addition, $\pi_1 X$ is co-Hopfian, then m is a homotopy equivalence.*

We generalize just as in the previous section.

THEOREM 9. *Let f: $X \to Y$, g: $Y \to X$ be monomorphisms in $\mathcal{H}$ and let X be Hft. Then Y is Hft and f factors as $X \xrightarrow{h} \tilde{Y} \xrightarrow{p} Y$, where h is a homotopy equivalence and p is a covering map.*

PROOF. We know that $g_* f_*$: $\pi_n X \cong \pi_n X$, $n \geqslant 2$ and g_*: $\pi_n Y \rightarrowtail \pi_n X$, $n \geqslant 1$, f_*: $\pi_n X \rightarrowtail \pi_n Y$, $n \geqslant 1$. Thus g_*: $\pi_n Y \cong \pi_n X$, $n \geqslant 2$, and f_*: $\pi_n X \cong \pi_n Y$, $n \geqslant 2$. We now lift f into $\tilde{Y}$, where $\pi_1 \tilde{Y} = f_* \pi_1 X$. The lifted map h must be a homotopy equivalence. Of course there is a similar factorization of g.

COROLLARY 10. *If, in addition, $\pi_1 X$ or $\pi_1 Y$ is co-Hopfian, then f and g are homotopy equivalences.*

We remark finally that the results of §§2 and 3 admit evident P-versions, where P is a family of primes, provided that we confine attention to nilpotent spaces (compare the Appendix of [**HR**]).

[2] The proof just given does not use the full force of the assumption that m is a monomorphism. Thus, if we were to use the fact that m_*: $\pi_n X \rightarrowtail \pi_n X$, $n \geqslant 2$, then it would suffice simply to establish that m_{*1}: $\pi_n X \otimes \mathbf{Z}/p \cong \pi_n X \otimes \mathbf{Z}/p$ for all p, $n \geqslant 2$.

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