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STABILITY OF TYPICAL CONTINUOUS FUNCTIONS WITH RESPECT TO SOME PROPERTIES OF THEIR ITERATES

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ABSTRACT. Let I be a real compact interval, and let C be the space of continuous functions $I \to I$ with the uniform metric. For $f \in C$ denote $\nu(f) = \sup_{x \in I}(\limsup_{n\to\infty} f^n(x) - \liminf_{n\to\infty} f^n(x))$, where f^n is the nth iterate of f. Then for each positive d there is an open set C^* dense in C such that the oscillation of ν at each point of C^* is less than d. Consequently, ν is continuous in C except of the points of a first Baire category set.

Let I be a compact real interval, C the metric space of continuous functions $I \to I$ with the uniform metric. For $f \in C$ let $\|f\| = \max\{f(x); x \in I\}$, and let f^n denote the nth iterate of f. If f has a cycle $x_1 \mapsto x_2 \mapsto \cdots \mapsto x_m \mapsto x_1$, where $x_i \neq x_j$ for $i \neq j$, $i, j = 1,\ldots,m$, then the order of this cycle is m while the width of this cycle is $\max(x_i - x_j)$. Let $\lambda(f)$ be the l.u.b. of the widths of all cycles of f, and put

$$\nu(f) = \sup_{x \in I}\Big(\limsup_{n\to\infty} f^n(x) - \liminf_{n\to\infty} f^n(x)\Big).$$

Clearly $0 \leqslant \lambda(f) \leqslant \nu(f)$ for each $f \in C$, and $\nu(f) = 0$ iff f has no cycles (cf. [**4**]). The function ν can be sometimes used as a "measure of chaos". Namely, if $f \in C$ has no cycles, then each neighbourhood of f contains a chaotic function (cf. [**2**]). But if additionally the set of fixed points of f contains no interval, then for each g sufficiently near to f, $\nu(g)$ is small (although it can be positive) (cf. [**5**]).

L. Block [**1**] recently has shown that continuous functions are stable with respect to the order of their cycles: if $f \in C$ has a cycle of order m, and if n is greater than m in the Šarkovskii ordering (cf. [**3** or **6**]), then each $g \in C$ sufficiently near to f has some n-cycle. However, Block's result gives no information on the width of the corresponding cycles. As we show (see Theorem 1) the width of the cycles of functions g from a certain neighbourhood of f can be essentially smaller then the width of cycles of f, i.e. $\lambda(g) \ll \lambda(f)$, and in application the cycles of g then cannot be distinguished from the noise. The main aim of this paper is to show that such a type of behaviour is untypical. We begin with the following example.

THEOREM 1. *Given a* $\delta > 0$ *there is a continuous* f: $[0,1] \to [0,1]$ *with the following properties*: $\lambda(f) = 1$ *and for each* $\varepsilon > 0$ *there is some continuous* g: $[0,1] \to [0,1]$ *such that* $\|f - g\| < \varepsilon$ *and* $\nu(g) < \delta$.

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PROOF. Let $I_1, I_2, \ldots, I_{2n+1}$ be pairwise disjoint closed subintervals of [0, 1] with the natural ordering

$$I_{2n+1} < I_{2n-1} < \cdots < I_1 < I_2 < I_4 < \cdots < I_{2n}$$

(i.e. $x < y$ for each $x \in I_3$ and $y \in I_1$, etc.). Put $I_i = [x_i, y_i]$ for each i, and assume $x_{2n+1} = 0$, $y_{2n} = 1$. Define a function f: $[0,1] \to [0,1]$ in the following way: $f(I_i) = I_{i+1}$ if $i \neq 2n+1$ and $f(I_{2n+1}) = I_1$. Moreover, f is linear and decreasing on each I_i where $i \neq 2n+1$, and linear and increasing on I_{2n+1}. Finally, extend f continuously onto the whole [0, 1].

It is easy to verify that f: $x_1 \mapsto y_2 \mapsto x_3 \mapsto y_4 \mapsto \cdots \mapsto y_{2n} \mapsto x_{2n+1} \mapsto x_1$, i.e. x_1 generates a cycle of order $(2n+1)$ and of the width 1. Similarly y_1 is a point of a $(2n+1)$-cycle, and from the linearity of f on each I_i we have f^{2n+1} is the identity mapping on I_1, and consequently, on each I_i. Thus each point of I_1 is a cyclic point of order $2n+1$.

Now define a continuous g: $[0,1] \to [0,1]$ such that $g(x) = f(x)$ for $x \in I_{2n+1}$, $g(x_{2n+1}) = x_1 + \varepsilon$, $g(y_{2n+1}) = y_1$, and g is linear on I_{2n+1}. Without loss of generality we may assume that $x_1 + \varepsilon < y_1$.

Clearly for each $x \in I_{2n+1}$, except of $x = y_{2n+1}$, we have $f(x) < g(x) = f(y)$ for some $y \in I_{2n+1}$, $y > x$. Hence

$$g^{2n+1}(x) = f^{2n+1}(y) = y > x. \tag{1}$$

Thus g has in I_{2n+1} exactly one cyclic point of order > 1, namely the point y_{2n+1}.

To finish the proof note that when the length of I_{2n+1} is greater than $1 - \delta$, then we have $\lambda(g) < \delta$. Moreover, by (1) the cycle of g generated by y_{2n+1} attracts all points of I_{2n+1}, and this implies $\nu(g) < \delta$. Q.E.D.

REMARK. The theorem holds when $2n+1$ is replaced by any $m > 1$. Also Theorem 1 shows that neither ν nor λ is lower semicontinuous. An example exhibiting that these functions are not upper semicontinuous can be found in [**5**].

In the proof of our main result the following lemmas are useful.

LEMMA 1. *Let $f \in C$ with $\nu(f) > d$. Then for each $\varepsilon > 0$ there is some $g \in C$ such that $\|f - g\| < \varepsilon$ and g has a cycle of the width greater than d.*

PROOF. Choose $x \in I$ such that

$$\limsup_{n\to\infty} f^n(x) - \liminf_{n\to\infty} f^n(x) > d. \tag{2}$$

For simplicity denote $f^n(x) = x_n$. Without loss of generality we may assume that the sequence $\{x_n\}$ is not periodic. Let $\delta > 0$ be such that $|f(u) - f(v)| < \varepsilon$ whenever $|u - v| < \delta$ for $u, v \in I$. By (2) there are indexes $n(1) < n(2) < n(3)$ such that

$$|x_{n(1)} - x_{n(3)}| < \delta \tag{3}$$

and

$$|x_{n(2)} - x_{n(3)}| > d. \tag{4}$$

Now let $g(x_n) = f(x_n)$ for $n = n(1)+1, \ldots, n(3)-1$, $g(x_{n(3)}) = f(x_{n(1)})$ and let g be continuous in I. Moreover, by (3), $|g(x_{n(3)}) - f(x_{n(3)})| < \varepsilon$ hence we can choose g

such that $\|f - g\| < \varepsilon$. It is easy to verify that g has a cycle $x_{n(1)+1} \mapsto x_{n(1)+2} \mapsto \cdots \mapsto x_{n(3)} \mapsto x_{n(1)+1}$ of order $k = n(3) - n(1)$. By (4) the width of this cycle is greater than d. Q.E.D.

LEMMA 2. *Assume that* $f \in C$ *and that* $\lambda(f) > d$. *Then for each* $\varepsilon > 0$ *there is some* $g \in C$ *with the following properties*: $\|f - g\| < \varepsilon$ *and for each* $h \in C$ *from a sufficiently small neighbourhood of* g *we have* $\lambda(h) > d$.

PROOF. Choose a cycle $x_1 \mapsto x_2 \mapsto \cdots \mapsto x_n \mapsto x_1$ of the function f of the width $d_1 > d$. First assume that one of the points x_i, say x_1, is an interior point of I. Choose also δ with $0 < \delta < (d_1 - d)/2$ such that for every $u, v \in I$,

$$|u - v| < \delta \quad \text{implies} \; |f(u) - f(v)| < \varepsilon. \tag{5}$$

Moreover, let U_1 be an open interval containing x_1 and denote $U_i = f^{i-1}(U_1)$, $i = 2, \ldots, n$. Without loss of generality we may assume that U_1 is so small that the sets U_i are pairwise disjoint, and that the length of U_1 is less than δ. Now choose $u, v \in U_1$ such that $u < x_1 < v$. Define g by $g(y) = f(y)$ for $y \notin U_1$, $g(u) = g(v) = x_2$, and extend g continuously to a function $I \to I$. By (5) g can be chosen such that $\|f - g\| < \varepsilon$. We have

$$g^n(u) = x_1 > u \quad \text{and} \quad g^n(v) = x_1 < v. \tag{6}$$

If $h \in C$ is near to g, then $h^i((u, v)) \subset U_{i+1}$, $i = 1, \ldots, n - 1$, and by (6), $h^n(u) > u$, $h^n(v) < v$. Hence h has a periodic point $\xi \in (u, v)$. The order of ξ must be n since $h^i(\xi) \in U_{i+1}$, $i = 1, \ldots, n - 1$. Also it is easy to see that the width of this cycle is greater than $d_1 - 2\delta > d$.

It remains to consider the case when $n = 2$ and x_1, x_2 are the endpoints of I. Assume that $x_1 < x_2$ and define g by $g(x) = x_2$ for $x \in [x_1, x_1 + \mu]$ and $g(x) = x_1$ for $x \in [x_2 - \mu, x_2]$, where $\mu > 0$ is small. Moreover, let $g(x) = f(x)$ for $x \in [x_1 + 2\mu, x_2 - 2\mu]$, and let g be continuous in I. Clearly for μ sufficiently small g can be chosen such that $\|f - g\| < \varepsilon$. Also assume that $x_2 - x_1 - 2\mu > d$. Now let $h \in C$ with $\|h - g\| < \mu/2$. Then $h^2(x_1 + \mu) = x_1 < x_1 + \mu$. On the other hand, since h: $I \to I$, we have $h(x_1) \geqslant x_1$. Thus for some $\xi \in [x_1, x_1 + \mu)$, $h^2(\xi) = \xi$ and clearly $h(\xi) \neq \xi$. It is easy to see that $|h(\xi) - \xi| > x_2 - x_1 - 2\mu > d$, i.e. $\lambda(h) > d$, and the lemma is proved.

In the following the oscillation $\omega_\nu(f)$ of ν at $f \in C$ is defined by

$$\omega_\nu(f) = \limsup |\nu(g) - \nu(f)| \quad \text{for} \; \|g - f\| \to 0, g \in C,$$

and similarly we define $\omega_\lambda(f)$. Now we are able to give the main result.

THEOREM 2. *Let* $\delta > 0$. *Then there is a subset* C_δ *of* C, *which is nowhere dense in* C *and such that the oscillation both of* ν *and* λ *at each* $f \in C \setminus C_\delta$ *is less than* δ.

In other words, when δ is small, both functions ν and λ are continuous in the points of the set $C \setminus C_\delta$ up to small perturbations.

COROLLARY. *There is a first Baire category set* $K \subset C$ *such that both* ν *and* λ *are continuous in the points of the set* $C \setminus K$.

PROOF. Put $K = \bigcup_{n=1}^{\infty} c_{1/n}$.

PROOF OF THEOREM 1. Fix some positive integer n. Let $A_i = \{f \in C;\ \nu(f) > i/n\}$ for $i = 1, \dots, n$, and put $A_0 = C$, $A_{n+1} = \varnothing$. Let B_i be the boundary of A_i, i.e. $B_i = \operatorname{Clos} A_i \cap \operatorname{Clos}(C \setminus A_i)$, where Clos is the closure operator in C. Then each B_i is closed and nowhere dense in C. To see the second property, let G be an open set. It suffices to consider the case $G \cap A_i \neq \varnothing$, where $i \neq 0$. By Lemma 1 there is some $f \in G$ with $\lambda(f) > i/n$, and by Lemma 2 there is a nonempty open subset $H \subset G$ such that $\lambda(h) > i/n$, and hence $\nu(h) > i/n$ for each $h \in H$. Thus $H \cap B_i = \varnothing$.

Now put $B = B_1 \cup \cdots \cup B_n$. Let $f \in C \setminus B$ and let j be the greatest index with $f \in A_j$. Then f is an interior point of A_j and f does not belong to the boundary of $A_{j+1} \subset A_j$; hence $f \in A_j \setminus \operatorname{Clos} A_{j+1}$. Now it is easy to see that $\omega_\nu(f) \leqslant 1/n$.

Similarly we can choose a nowhere dense set $D \subset C$ such that $\omega_\lambda(f) \leqslant 1/n$ for each $f \in C \setminus D$. To finish the proof take n such that $1/n < \delta$ and put $C_\delta = B \cup D$.

REMARK. There is an open problem, whether for some f, $\lambda(f) < \nu(f)$. We conjecture that the answer is positive, but we have no example of such a function.

REFERENCES

1. L. Block, *Stability of periodic orbits in the theorem of Šarkovskii*, Proc. Amer. Math. Soc. **81** (1981), 333–336.
2. P. E. Kloeden, *Chaotic difference equations are dense*, Bull. Austral. Math. Soc. **15** (1976), 371–379.
3. A. N. Šarkovskii, *Coexistence of cycles of a continuous transformation of a line into itself*, Ukrain. Mat. Ž. **16** (1964), no. 1, 61–71. (Russian)
4. ______, *On cycles and the structure of a continous transformation*, Ukrain. Mat. Ž. **17** (1965), no. 3, 104–111.
5. J. Smítal and K. Smítalová, *Structural stability of non-chaotic difference equations*, J. Math. Anal. Appl. **89** (1982).
6. P. Štefan, *A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys. **54** (1977), 237–248.

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