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CLOSED MAPS AND THE CHARACTER OF SPACES

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ABSTRACT. We give some necessary and sufficient conditions for the character of spaces to be preserved under closed maps.

Introduction. Recall the following result due to K. Morita and S. Hanai [**11**] or A. H. Stone [**14**]: Let Y be the image of a metric space X under a closed map f. Then Y is first countable (or metrizable) if and only if every boundary $\partial f^{-1}(y)$ of the point-inverse $f^{-1}(y)$ is compact.

E. Michael [**8**] showed that every $\partial f^{-1}(y)$ is compact if X is paracompact, and Y is locally compact or first countable. For the Lindelöfness of $\partial f^{-1}(y)$ with X metric, see [**15**].

For a space X and $x \in X$, let $\chi(x, X)$ be the smallest cardinal number of the form $|\mathfrak{B}(x)|$, where $\mathfrak{B}(x)$ is a nbd base at x in X. The character $\chi(X)$ of X is defined as the supremum of all numbers $\chi(x, X)$ for $x \in X$. Let f be a closed map from X onto Y. First, we show that the character of Y has an influence on the boundaries $\partial f^{-1}(y)$; indeed, they become Lindelöf or α-compact by the situation of Y. Second, in terms of these boundaries, we give some necessary and sufficient conditions for the character of X to be preserved under f.

We assume all spaces are regular and all maps are continuous and onto.

1. α-compactness of the boundaries. We recall some definitions. A space X is *strongly collectionwise Hausdorff* if, whenever $D = \{x_\alpha;\ \alpha \in A\}$ is a discrete closed subset of X, there is a discrete collection $\{U_\alpha;\ \alpha \in A\}$ of open subsets with $U_\alpha \cap D = \{x_\alpha\}$. Every paracompact space is strongly collectionwise Hausdorff.

Let $\alpha \geqslant \omega_0$ and α^+ be the least cardinal number greater than α. A space X is *α-compact* if every subset of X of cardinality α has an accumulation point in X. A space X is *α-Lindelöf* if every open cover of X has a subcover of cardinality $\leqslant \alpha$. Every α-Lindelöf space is α^+-compact.

A space X is *sequential* if $F \subset X$ is closed in X whenever $F \cap C$ is closed in C for each compact metric subset C of X. If we replace "compact metric subset" by "countable subset", then such a space is said to have *countable tightness*. Every sequential space is precisely the quotient image of a metric space [**3**].

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Following the terminology "c-sequential spaces" defined in [**13**], let us call $x \in X$ a *c-sequential point* in X if, whenever x is not isolated in a closed subset F of X, there exists a sequence in $F - \{x\}$ converging to the point x. Every point of a sequential space is c-sequential.

PROPOSITION 1.1. *Let f: $X \to Y$ be a closed map with X strongly collectionwise Hausdorff. If $y \in Y$ satisfies* (1) *or* (2) *below, then $\partial f^{-1}(y)$ is α-compact.*

(1) *The point y has a nbd system $\{V_\beta;\ \beta < \alpha\}$ such that if $y_\beta \in V_\beta$ and the y_β are all distinct, then $\{y_\beta;\ \beta < \alpha\}$ has an accumulation point in Y.*

(2) *The point y is c-sequential and $\chi(y, Y) < 2^\alpha$.*

PROOF. Suppose $\partial f^{-1}(y)$ is not α-compact. Then there exists a discrete collection $\{U_\beta;\ \beta < \alpha\}$ of open subsets of X meeting $\partial f^{-1}(y)$.

Case (1). There exists $x_0 \in U_0 \cap (f^{-1}(V_0) - f^{-1}(y))$; hence $x_0 \in U_0$ and $f(x_0) \in V_0 - \{y\}$. For $\beta < \alpha$, assume there exists a subset $F_\beta = \{f(x_\gamma);\ \gamma < \beta\}$ of Y such that $x_\gamma \in U_\gamma$, $f(x_\gamma) \in V_\gamma - \{y\}$, and the $f(x_\gamma)$ are all distinct. Then, since F_β is closed in Y with $F_\beta \not\ni y$, there exists a nbd V of y with $V \cap F_\beta = \varnothing$. Thus there exists $x_\beta \in U_\beta \cap (f^{-1}(V_\beta \cap V) - f^{-1}(y))$, hence $x_\beta \in U_\beta$, $f(x_\beta) \in V_\beta - \{y\}$, and $f(x_\beta) \notin F_\beta$. Then, by induction there exists a subset $F = \{f(x_\beta);\ \beta < \alpha\}$ of Y such that $x_\beta \in U_\beta$, $f(x_\beta) \in V_\beta$ and the $f(x_\beta)$ are all distinct. Thus, F is discrete in Y, but it has an accumulation point in Y. This is a contradiction. Hence $\partial f^{-1}(y)$ is α-compact.

Case (2). Since the point y is not isolated in $f(\overline{U_\beta})$ for each $\beta < \alpha$, there exist sequences C_β in $f(\overline{U_\beta}) - \{y\}$ converging to y. For $\gamma < \alpha$, assume there exists a pairwise disjoint collection $\{C_{\beta(\delta)};\ \delta < \gamma\}$ in Y such that each $C_{\beta(\delta)}$ is some C_β except at finitely many points. Since $\{C_{\beta(\delta)};\ \delta < \gamma\}$ is hereditarily closure-preserving, for each $\delta < \gamma$, $\{\beta;\ C_\beta \cap C_{\beta(\delta)}$ is infinite$\}$ is finite. Thus, there exists $\beta_0 < \alpha$ such that $C_{\beta_0} \cap C_{\beta(\delta)}$ is finite for each $\delta < \gamma$. Then it follows that $C_{\beta_0} \cap \bigcup_{\delta<\gamma} C_{\beta(\delta)}$ is finite. Put $C_{\beta(\gamma)} = C_{\beta_0} - \bigcup_{\delta<\gamma} C_{\beta(\delta)}$. Then $C_{\beta(\gamma)} \cap C_{\beta(\delta)} = \varnothing$ for each $\delta < \gamma$. Hence, by induction there is a pairwise disjoint collection $\{C_{\beta(\gamma)};\ \gamma < \alpha\}$ in Y such that each $C_{\beta(\gamma)}$ is assumed to be some C_β. Let $S = \bigcup_{\gamma<\alpha}(C_{\beta(\gamma)} \cup \{y\})$. But, since $\{C_{\beta(\gamma)} \cup \{y\};\ \gamma < \alpha\}$ is a hereditarily-closure preserving closed cover of S, it is easy to show that $U \subset S$ is open (resp. closed) in S whenever $U \cap (C_{\beta(\gamma)} \cup \{y\})$ is open (resp. closed) in $C_{\beta(\gamma)} \cup \{y\}$ for each $\gamma < \alpha$. Then $\chi(y, S) = 2^\alpha$; hence $\chi(y, Y) \geqslant 2^\alpha$. This is a contradiction. Hence, $\partial f^{-1}(y)$ is α-compact.

COROLLARY 1.2. *Let f: $X \to Y$ be a closed map with X strongly collectionwise Hausdorff. Then $\partial f^{-1}(y)$ is α-compact if one of the following three properties is satisfied.*

(1) *y has a nbd which is α-compact.*

(2) $\chi(y, Y) \leqslant \alpha$.

(3) *Y is sequential and $\chi(y, Y) < 2^\alpha$.*

Under Martin's Axiom (MA), using [**7**, Proposition 1.1] we have the following by Proposition 1.1 (Case (2)).

COROLLARY 1.3 (MA). *Let $f\colon X \to Y$ be a closed map with X normal. If Y has countable tightness, then $\partial f^{-1}(y)$ is countably compact if $\chi(y, Y) < 2^{\omega_0}$. When Y is especially sequential, we can omit the assumption (MA).*

In terms of certain quotient ranges, we give another sufficient condition for the boundaries to be α^+-compact (indeed, α-Lindelöf). First, we state definitions.

A space X is *inner-one A* [**10**] if, whenever (A_i) is a decreasing sequence of subsets with $x \in \overline{A_i - \{x\}}$ (denoted by $(A_i)\downarrow x$), then there exist $a_i \in A_i$ such that $\{a_i;\, i \in \omega_0\}$ is not closed in X. Every q-space [**8**], or more generally every countably bi-quasi-k-space [**9**], is inner-one A. Recall that a space X is *perfect* if every closed subset of X is a G_δ-set.

PROPOSITION 1.4. *Let $f\colon X \to Z$ be a closed map with X paracompact and Z sequential. Let $g\colon Y \to Z$ be a quotient map. Then $\partial f^{-1}(z)$ is α-Lindelöf if $\partial g^{-1}(z)$ is α-Lindelöf, and* (1) *or* (2) *below holds.*

(1) $(2^\alpha < 2^{\alpha^+})$. *$\chi(Y) \leqslant 2^\alpha$, and either Y is sequential or perfect.*

(2) *Y is inner-one A.*

PROOF. Suppose $\partial f^{-1}(z)$ is not α-Lindelöf. Since $\partial f^{-1}(z)$ is paracompact, $\partial f^{-1}(z)$ is not α^+-compact. Since Z is sequential, by the proof of Proposition 1.1, Z contains a closed copy S of the space obtained from the disjoint union of convergent sequences $\{C_\beta;\, \beta < \alpha^+\}$ by identifying all the limit points to the point z. Let $T = g^{-1}(S)$ and $h = g\,|\,T$. Since S is closed in Z, h is a quotient map.

Case (1). The space S is Fréchet; that is, if $s \in \overline{A}$ in S, then there exists a sequence in A converging to the point s. Hence, by the proof of [**3**, Proposition 2.3], the quotient map h onto S is a pseudo-open map (stated in the first paragraph of the next section). The closed subset T of Y is a sequential or perfect space with $\chi(T) \leqslant 2^\alpha$, and a closed subset $\partial_T h^{-1}(z)$ of $\partial_Y g^{-1}(z)$ is α-Lindelöf. Thus, under $(2^\alpha < 2^{\alpha^+})$ we have $\chi(z, S) < 2^{\alpha^+}$ by Proposition 2.4 in the next section. This is a contradiction. Hence, $\partial f^{-1}(z)$ is α-Lindelöf.

Case (2). Assume the convergent sequences C_β are subsets of S (hence, $C_\gamma \cap C_\delta = \{z\}$ if $\gamma \neq \delta$), and let $T_\beta = h^{-1}(C_\beta - \{z\})$ for each $\beta < \alpha^+$. Since each T_β is not closed in T, there exists a subset $T^* = \{t_\beta;\, \beta < \alpha^+\}$ of T with $t_\beta \in \overline{T_\beta} - T_\beta$. Suppose $|T^*| < \alpha^+$. Then there exists $\{\beta_i;\, i \in \omega_0\}$ such that the t_{β_i} are all the same point. Let $A_i = \bigcup_{j \geqslant i} T_{\beta_j}$ for each i. Then $(A_i)\downarrow t_{\beta_0}$.

Since T is inner-one A, there exist $a_i \in A_i$ such that $A = \{a_i;\, i \in \omega_0\}$ is not closed in T. Let $H_i = h^{-1}(C_{\beta_i}) \cap A$ for each i. Then each $h(H_i)$ is a finite subset of C_{β_i} with $h(H_i) \not\ni z$. Let $V = \bigcup_{i \in \omega_0}(C_{\beta_i} - h(H_i))$. Then $h^{-1}(V) \cap A = \varnothing$ and V is a nbd of the point z in $S' = \bigcup_{i \in \omega_0} C_{\beta_i} \subset S$. Then A is closed in $h^{-1}(S')$, hence in T. This contradiction implies the set T^* has cardinality α^+. Since $\partial h^{-1}(z)$ is an α^+-compact subset of T which contains the set T^*, there exists a point $t' \in T$ and a subset T' of T^* accumulating to the point t'. Since T is inner-one A, there exists a point $t \in T$ and a subset $\{b_i;\, i \in \omega_0\}$ of T' accumulating to the point t. Let $b_i = t_{\beta_i'}$ and $A_i' = \bigcup_{j \geqslant i} h^{-1}(C_{\beta_j'} - \{z\})$ for each i. Then $(A_i')\downarrow t$. However, we have a contradiction by the same way as in the case where $|T^*| < \alpha^+$. Thus, $\partial f^{-1}(z)$ is α-Lindelöf.

Since every quotient image of a sequential space is obviously sequential, we have

COROLLARY 1.5. ($2^\alpha < 2^{\alpha^+}$). *Let $f\colon X \to Z$ be a closed map with X paracompact, and let $g\colon Y \to Z$ be a quotient map. Then $\partial f^{-1}(z)$ is α-Lindelöf if $\partial g^{-1}(z)$ is also, and Y is a sequential space with $\chi(Y) \leqslant 2^\alpha$. When Y is especially first countable, we can omit the assumption ($2^\alpha < 2^{\alpha^+}$). Furthermore, we can replace "α-Lindelöf" by "α^+-compact" if we replace paracompactness by strongly collectionwise Hausdorfness.*

COROLLARY 1.6. *Let $f_i\colon X_i \to Y$ ($i = 1, 2$) be closed maps with X_i paracompact first countable. Then $\partial f_1^{-1}(y)$ is Lindelöf and compact if and only if $\partial f_2^{-1}(y)$ is, respectively.*

2. Preservation of the character. A map $f\colon X \to Y$ is *pseudo-open* [**1**] if for any $y \in Y$ and any nbd U of $f^{-1}(y)$, $y \in \operatorname{int} f(U)$; equivalently, f is hereditarily quotient, that is, $f|f^{-1}(S)$ is quotient for each $S \subset Y$ by [**1**, Theorem 1]. Every closed map or every open map is pseudo-open.

LEMMA 2.1. *Let $f\colon X \to Y$ be a pseudo-open map with $\chi(X) \leqslant 2^\alpha$. Then $\chi(y, Y) \leqslant 2^\alpha$ if $\partial f^{-1}(y)$ is an α-Lindelöf space of cardinality $\leqslant 2^\alpha$.*

PROOF. Since $\chi(X) \leqslant 2^\alpha$ and $|\partial f^{-1}(y)| \leqslant 2^\alpha$, there is an open collection $\mathfrak{B}$ of cardinality $\leqslant 2^\alpha$ such that for $x \in \partial f^{-1}(y)$ and a nbd V of x in X, $x \in B \subset V$ for some $B \in \mathfrak{B}$. Let U be a nbd of y in Y. Since $\partial f^{-1}(y) \subset f^{-1}(U)$ and $\partial f^{-1}(y)$ is α-Lindelöf, there is a subfamily $\mathfrak{B}'$ of $\mathfrak{B}$ with $|\mathfrak{B}'| \leqslant \alpha$ such that $\partial f^{-1}(y) \subset \bigcup \mathfrak{B}' \subset f^{-1}(U)$. Thus, $y \in \operatorname{int} f(\bigcup \mathfrak{B}') \cup f(\operatorname{int} f^{-1}(y)) \subset U$. Hence, $\chi(y, Y) \leqslant 2^\alpha$.

EXAMPLE 2.2. In Lemma 2.1, that f is pseudo-open is essential. Indeed, let $X_1 = D \cup \{\infty\}$ be the one-point compactification of a discrete space D of cardinality α. For each $d \in D$, let $I_d = [0, 1] \times \{d\}$ be a copy of the closed unit interval $[0, 1]$, and let X_2 be the disjoint union of $\{I_d;\ d \in D\}$. Then the space Y obtained from the disjoint union X of X_1 and X_2 by identifying $d \in X_1$ to $(1, d) \in X_2$ for each $d \in D$ is the quotient finite-to-one image of a paracompact space X with $\chi(X) = \alpha$, but $\chi(\infty, Y) = 2^\alpha$.

The following lemma is due to Juhász [**5**] (cf. [**4**]) for Case (1), and Arhangel'skiĭ [**2**] for Case (2).

LEMMA 2.3. *Let X satisfy* (1) *or* (2) *below. Then $|X| \leqslant 2^\alpha$.*

(1) *Hereditarily α-Lindelöf space.*

(2) *Sequential α-Lindelöf space of character $\leqslant 2^\alpha$.*

Every α-Lindelöf perfect space is obviously hereditarily α-Lindelöf. Thus, by Lemmas 2.1 and 2.3, we have

PROPOSITION 2.4. *Let $f\colon X \to Y$ be a pseudo-open map with $\chi(X) \leqslant 2^\alpha$. Then $\chi(y, Y) \leqslant 2^\alpha$ if $\partial f^{-1}(y)$ is α-Lindelöf, and either X (or $\partial f^{-1}(y)$) is sequential or perfect.*

The following example[1] shows that the sequentiality or perfectness of X (or $\partial f^{-1}(y)$) in the previous proposition is essential even if f is perfect.

[1] This was suggested by G. Gruenhage.

EXAMPLE 2.5. For each $\alpha \geqslant \omega_0$, there is a perfect map f: $X \to Y$ such that X is a paracompact space with $\chi(X) = \alpha$ and Y is a Fréchet space with $\chi(Y) = 2^\alpha$. Indeed, let $K_i = K \times \{i\}$ $(i = 0, 1)$ be copies of a compact space K with $|K| = 2^\alpha$ and $\chi(K) = \alpha$ (e.g., let $K = I^\alpha$, where I is the closed unit interval). Let $X = K_0 \cup K_1$ be the Alexandorff Double of K; that is, each point of K_1 is isolated, and $V \times \{0, 1\} - \{(x, 1)\}$ is a nbd of $(x, 0)$ in K_0, where V is a nbd of x in K. Then the space Y, obtained from X by identifying K_0 to a single point, is the closed image of a compact space X with $\chi(X) = \alpha$. But Y is the one-point compactification of a discrete space of cardinality 2^α. Thus, Y is a Fréchet space with $\chi(Y) = 2^\alpha$.

REMARK 2.6. Not every perfect image of a first countable space X is first countable even if X is a compact space (in the above example, put $K = I$), or a σ-space (e.g., see [**6**, Example 4.3]). Thus, in Proposition 2.4, we cannot replace "2^α" by "ω_0", even if f is a perfect map.

Every closed image of a sequential space is sequential, and every α^+-compact paracompact space is α-Lindelöf. Thus, combining Proposition 2.4 with Corollary 1.2 (Case (3)), we have

THEOREM 2.7. $(2^\alpha < 2^{\alpha^+})$. *Let f: $X \to Y$ be a closed map with X a paracompact sequential space with $\chi(X) \leqslant 2^\alpha$. Then every $\partial f^{-1}(y)$ is α-Lindelöf if and only if $\chi(Y) \leqslant 2^\alpha$.*

COROLLARY 2.8. $(2^\alpha < 2^{\alpha^+})$. *Let f: $X \to Y$ be a closed map with X a paracompact first countable space. Then every $\partial f^{-1}(y)$ is α-Lindelöf if and only if $\chi(Y) \leqslant 2^\alpha$.*

As a generalization of σ-spaces and paracompact M-spaces, K. Nagami [**12**] defined strong Σ-spaces (he also defined Σ-spaces). For the definition of strong Σ-spaces, see Definition 1.1 (or Lemma 1.4) in [**12**].

LEMMA 2.9. *Let α be a cardinal number with* $\mathrm{cf}(\alpha) > \omega_0$. *Then every α-compact strong Σ-space is β-Lindelöf for some $\beta < \alpha$.*

PROOF. Let X be a strong Σ-space. Then there exists a cover $\mathcal{K}$ of compact subsets of X and a Σ-net $\mathcal{F} = \bigcup_{i \in \omega_0} \mathcal{F}_i$ for X such that each $\mathcal{F}_i$ is a locally finite closed cover of X, and if $K \subset U$ with $K \in \mathcal{K}$ and U open, then $K \subset F \subset U$ for some $F \in \mathcal{F}$. If X is α-compact, by $\mathrm{cf}(\alpha) > \omega_0$, the Σ-net $\mathcal{F}$ has cardinality $\leqslant \beta$ for some $\beta < \alpha$. Then it is easy to show that X is β-Lindelöf.

THEOREM 2.10 (MA). *Let f: $X \to Y$ be a closed map with X a paracompact Σ-space with $\chi(X) \leqslant 2^{\omega_0}$. Suppose X is sequential or perfect. Then every $\partial f^{-1}(y)$ is 2^{ω_0}-compact if and only if $\chi(Y) \leqslant 2^{\omega_0}$.*

PROOF. The "if" part follows from Corollary 1.2 (Case (2)).

"Only if": Since every $\partial f^{-1}(y)$ is a strong Σ-space, by Lemma 2.9, $\partial f^{-1}(y)$ is β-Lindelöf for some $\beta < 2^{\omega_0}$. We remark that every β-Lindelöf perfect space is hereditarily β-Lindelöf. Thus, since $\chi(X) \leqslant 2^{\omega_0} = 2^\beta$, $\chi(Y) \leqslant 2^{\omega_0}$ by Proposition 2.4.

COROLLARY 2.11 (MA). *Let $f\colon X \to Y$ be a closed map with X a paracompact space with $\chi(X) \leqslant 2^{\omega_0}$. Suppose X is a σ-space or a sequential M-space. Then every $\partial f^{-1}(y)$ is 2^{ω_0}-compact if and only if $\chi(Y) \leqslant 2^{\omega_0}$.*

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