

A STRONG FORM OF THE PHRAGMEN-BROUWER THEOREM

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ABSTRACT. In this paper we prove the following form of the Phragmen-Brouwer Theorem: a locally connected, connected normal T_1 -space X is unicoherent if and only if for every pair of disjoint nonseparating continua C and D in X , $C \cup D$ does not separate X . Among the several corollaries is the proposition: X is multicoherent if and only if X is the union of a circular chain of continua $\{A_0, A_1, A_2, A_3\}$ where no three of the A_i 's have a point in common.

1. Introduction. Throughout this paper X will denote a normal, locally connected, connected T_1 -space. By a *continuum* we mean a closed and connected (not necessarily compact) subset of X , a *region* is an open connected subset of X . For $A \subset X$, $b_0(A)$ denotes the number of components of A less one (or ∞ if this number is infinite). The *degree of multicoherence*, $r(X)$, of X is defined by

$$r(X) = \sup\{b_0(H \cap K) : X = H \cup K \text{ and } H \text{ and } K \text{ are subcontinua of } X\}.$$

If $r(X) = 0$, X is said to be *unicoherent* and we say that X is *multicoherent* otherwise. If $0 < r(X) < \infty$, we say that X is *finitely multicoherent* and if $0 < r(X) \leq \infty$ but $b_0(H \cap K) < \infty$ for any representation $X = H \cup K$, where H and K are continua, we say that X is *weakly-finite multicoherent*.

A connected subset C of X is said to be *simple* if $X \setminus C$ is also connected.

Unicoherence is characterized in the Phragmen-Brouwer Theorem (PB Theorem) as follows (see [10, Theorem (3.2)]):

A connected, locally connected, normal space X is unicoherent if and only if whenever A and B are disjoint closed sets and neither A nor B separates X , then $A \cup B$ does not separate X .

J. H. V. Hunt extended the PB Theorem to arbitrary locally connected topological spaces in [7], and he and E. D. Tymchatyn were able to replace "closed" by "arbitrary set" in [8]. The PB Theorem has been generalized by A. H. Stone in [11, Theorem 5 and 13, Theorem 6] to multicoherent spaces, i.e. to spaces X , where $1 \leq r(X)$.

The principal result of this paper is the following:

A connected, locally connected normal T_1 -space X is unicoherent if and only if whenever C and D are disjoint simple continua in X , $X \setminus (C \cup D)$ is connected.

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This proposition was known to hold for compact spaces or weakly-finitely multicoherent spaces [5]; in [4], a proof for this proposition, when X is assumed to be perfectly normal, was given.

We call the proposition above “a strong form” of the PB Theorem because in the sufficiency (i.e. the if part), we have weakened the hypothesis to requiring that A and B be continua, while essentially retaining the general setting (connected, locally connected and normal) of the PB Theorem. The reader is referred to [7] for additional comments on the nomenclature of the “Phragmen-Brouwer Theorem”.

2. Preliminary results.

DEFINITIONS AND NOTATION. Let A and B be nonempty, disjoint closed subsets of X . A *division* of X with respect to A and B is a representation $X = R \cup F \cup S$, where R and S are disjoint regions, $\text{Fr } R = \text{Fr } S$, $\text{Fr } R \subseteq F$, $\text{Fr } R \cap (A \cup B) = \emptyset$ and $A \cap R \neq \emptyset \neq B \cap S$ (cf. [14, p. 49]).

LEMMA (2.1). *Let A and B be disjoint closed subsets of X and let $a \in A$ and $b \in B$. Then there exists a division $X = R \cup F \cup S$ such that $a \in R$, $b \in S$, and R and S are simple regions in X . Furthermore, if A (respectively, B) is a continuum, then $A \cap F = \emptyset$ (respectively, $B \cap F = \emptyset$).*

PROOF. Let U be an open subset of X such that $A \subseteq U \subseteq \bar{U} \subseteq X \setminus B$, let P be the component of U containing a , and let S be the component of $X \setminus \bar{P}$ containing b and let $F = X \setminus (R \cup S)$. Then since X is locally connected, $\text{Fr } R = \text{Fr } S \subseteq \text{Fr } U \subseteq X \setminus (A \cup B)$. Also, since X is connected, every component of $X \setminus (\bar{R} \cup \bar{S})$ has a limit point in $\text{Fr } R$. This implies that $X \setminus S = R \cup F$ and $X \setminus R = S \cup F$ are connected sets, as required. Finally, if A is a continuum, $A \subseteq R$ since $A \cap R \neq \emptyset$ and $A \cap \text{Fr } R = \emptyset$; hence $A \cap F = \emptyset$. This completes the proof of Lemma (2.1).

3. Main result.

THEOREM (3.1) (STRONG FORM OF THE PHRAGMEN-BROUWER THEOREM). *A connected, locally connected normal T_1 -space is unicoherent if (and only if) whenever C and D are disjoint simple continua in X , then $X/(C \cup D)$ is connected.*

PROOF. Suppose that if neither of two disjoint continua separate X , then their union does not separate X and suppose that X is not unicoherent. Then there exist continua H and K such that $X = H \cup K$ and $H \cup K = A \cup B$ where A and B are nonempty disjoint closed sets. Let $a \in A$ and $b \in B$. By Lemma (2.1) there exists a division $X = R \cup F \cup S$ where $a \in R$, $b \in S$ and $\text{Fr } R \cap (A \cup B) = \emptyset$.

Let $\mathcal{C} = \{\bar{S} \cap \bar{R}' : X = R' \cup F' \cup S' \text{ is a division of } X, \bar{R} \subseteq R', b \in S' \text{ and } \text{Fr } R' \cap (A \cup B) = \emptyset\}$. We assert that $\bigcap \mathcal{C} = \text{Fr } S$. It is clear that $\text{Fr } S \subseteq \bigcap \mathcal{C}$. Suppose $x \notin \text{Fr } S$; we will argue that $x \notin \bigcap \mathcal{C}$. Since $x \notin \text{Fr } S$, $x \in (X \setminus \bar{S})$ or $x \in S$. If $x \in X \setminus \bar{S}$, then $X \setminus \bar{S}$ is an open set containing x that misses $\bigcap \mathcal{C}$. Hence, in the first case $x \notin \bigcap \mathcal{C}$. On the other hand suppose $x \in S$. Then since X is a locally connected normal T_1 -space, there is a continuum B_0 containing $\{x, b\}$ and such that $B_0 \subseteq S$. Then by Lemma (2.1), there is a division $X = R'' \cup F'' \cup S''$ such that $\bar{R} \subseteq R''$, $B_0 \subseteq S''$ and $\text{Fr } R'' \cap (A \cup B \cup B_0) = \emptyset$. Then $x \in S''$ and $S'' \cap (\bar{S} \cap \bar{R}') = \emptyset$. Hence S'' misses $\bigcap \mathcal{C}$ and so $\text{Fr } S = \bigcap \mathcal{C}$.

Let βX denote the Stone-Ćech compactification of X and $\text{cl}_\beta S$ denote the closure in βX of a subset S of βX . Let $T = \text{cl}_\beta(A \cup B)$ and let $D = \text{cl}_\beta \text{Fr} S$. Since $\text{Fr} S \cap (A \cup B) = \emptyset$, $\text{Fr} S$ and $A \cup B$ are completely separated subsets of X and $D \cap T = \emptyset$ [9, p. 86]. Now D contains $\text{Fr} S$ and this implies that D meets each of the disjoint open sets $U = (\text{cl}_\beta H) \setminus (T)$ and $V = (\text{cl}_\beta K) \setminus T$. Since $\beta X \setminus (U \cup V)$ is compact, there exist divisions $X = R_i \cup F_i \cup S_i$, $i = 1, \dots, n$, such that $\bar{R} \subseteq R_i$, $b \in S_i$ and $\text{Fr} R_i \cap (A \cup B) = \emptyset$ and $\beta X \setminus (U \cup V) \subseteq \bigcup_{i=1}^n \{\beta X \setminus \text{cl}_\beta(S \cap R_i)\}$.

Since $b \in \bigcap_{i=1}^n S_i$, $\bigcup_{i=1}^n \bar{S}_i$ is a continuum and by Lemma (2.1) there exists a division $X = R_0 \cup F_0 \cup S_0$ such that $\bar{R} \subseteq R_0$, $\bigcup_{i=1}^n \bar{S}_i \subseteq S_0$ and $\text{Fr} R_0 \cap (A \cup B) = \emptyset$. It then follows that $\text{cl}_\beta(S \cap R_0) \subseteq \bigcap_{i=1}^n \text{cl}_\beta(S \cap R_i)$ and so $\beta X \setminus (U \cup V) \subseteq \beta X \setminus \text{cl}_\beta(S \cap R_0)$. By our hypothesis $S \cap R_0 = [X \setminus (R \cup F)] \cap [X \setminus (S_0 \cup F_0)] = X \setminus [(R \cup F) \cup (S_0 \cup F_0)]$ is a connected set as in $\text{cl}_\beta(S \cap R_0)$. Of course this means the connected set $\text{cl}_\beta(S \cap R_0)$ meets each of U and V and thus $\text{cl}_\beta(S \cap R_0)$ cannot be a subset of $U \cup V$. This contradiction implies that X must be unicoherent and this completes the proof.

REMARK. Note that in the proof of Theorem (3.1), the closedness of the points of X was employed twice: first in the continuumwise connectivity of the region S so as to obtain the subcontinuum B_0 above, and secondly in the existence of βX . In regard to the latter use, we could have employed the well-known construction of the Tychonoff space Y and the continuous surjection $\sigma: X \rightarrow Y$, where the correspondence $f \rightarrow f \circ \sigma$ is an isomorphism of $C(Y)$ onto $C(X)$, and avoided the necessity of assuming points are closed. However, the author has not been able to obtain a comparable substitution in the first use of T_1 -ness above.

The following proposition follows from the above theorem and [6, Theorem 1 and 5, Theorem 2], albeit via a rather circuitous route. A proof is provided herein for completeness.

COROLLARY (3.2). *The following are equivalent:*

- (i) X is multicoherent.
- (ii) There exist simple continua A_0, A_1, A_2 and A_3 such that
 - (a) $X = A_0 \cup A_1 \cup A_2 \cup A_3$,
 - (b) no three of the A 's have a point in common,
 - (c) $A_i \cap A_j \neq \emptyset$ if and only if $|(i - j) \bmod 4| \leq 1$.

(iii) There exist nonempty disjoint simple continua B_1 and B_2 such that $B_1 \cup B_2$ separates X .

PROOF. We shall show (i) implies (ii). Suppose X is multicoherent. Then, by Theorem (3.1), there exist simple continua C and D such that $X \setminus (C \cup D) = P \cup Q$ where P and Q are nonempty disjoint open sets. By Lemma (2.1) there exist divisions $X = R_i \cup F_i \cup S_i$, $0 \leq i \leq 2$, such that $C \subseteq R_0 \cap R_1 \cap R_2$, $D \subseteq S_0 \cap S_1 \cap S_2$, $(R_1 \cup F_1) \subseteq R_0 \subseteq (R_0 \cup F_0) \subseteq R_2$.

Since C and D are simple sets and since $R_1 \cup F_1$ and $S_1 \cup F_2$ are simple sets, every component of

$$P \cap [X \setminus (R_1 \cup F_1 \cup S_2 \cup F_2)] \quad \text{and} \quad Q \cap [X \setminus (R_1 \cup F_1 \cup S_2 \cup F_2)]$$

has a limit point in each of \bar{R}_1 and \bar{S}_2 . Similarly, we obtain: each of $A_0 = R_1 \cup F_1 \cup [P \cap (R_0 \cup F_0)]$, $A_1 = S_2 \cup F_2 \cup [P \cap (S_0 \cup F_0)]$, $A_2 = S_2 \cup F_2 \cup [Q \cap (S_0 \cup F_0)]$ and $A_3 = R_1 \cup F_1 \cup [Q \cap (R_0 \cup F_0)]$ are simple continua in X . Then $A_0 \cap A_1 = P \cap F_0$, $A_2 \cap A_3 = Q \cap F_0$, $A_0 \cap A_3 = R_1 \cup F_1$ and $A_1 \cap A_2 = S_2 \cap F_2$ and thus no three of the A_i 's have a point in common. This completes the proof of (i) implies (ii).

Now (i) implies (iii) follows from the above if we let $B_1 = A_0 \cap A_3$ and $B_2 = A_1 \cap A_2$ since $B_1 = R_1 \cap F_1$ and $B_2 = S_2 \cap F_2$ are simple continua in X and $B_1 \cup B_2$ separates X . Furthermore, (iii) implies (ii), by the proof of (i) implies (ii), for in it we merely assumed there existed disjoint simple continua, say B_1 and B_2 (or C and D), whose union separates X .

Hence, it remains to show that (ii) implies (i). But this is easy, for if $L = A_0 \cup A_1$ and $M = A_2 \cup A_3$, L and M are continua and $L \cap M$ is a separated closed set. Thus X is multicoherent and this completes the proof of the corollary.

REMARK (3.3). Note that Corollary (3.2) has Theorem (3.1) as an easy consequence: For suppose every pair of disjoint continua in X can be separated by a continuum and X is not unicoherent. Then by Corollary (3.2), $X = A_0 \cup A_1 \cup A_2 \cup A_3$ as in (3.2)(ii). But A_0 and A_3 cannot be separated by a continuum, so X must be unicoherent.

REMARK (3.4). The circularity, $m(X)$, of X is defined by: $m(X) = \sup\{n \in \mathbf{N}: X = \bigcup \{A_i: 0 \leq i < n - 1 \text{ where each } A_i \text{ is a nonempty continuum, no three } A_i\text{'s have a point in common, and } A_i \cap A_j \neq \emptyset \text{ if and only if } |(i - j) \bmod n| \leq 1\}\}$. A. H. Stone showed that $m(X) \geq 3$ if X is multicoherent [10]; it is known that $m(X) \geq 6$ if X is multicoherent and compact, and $m(X) = \infty$ if X is weakly-finitely multicoherent [5]. Furthermore, if X is a planar, Peano continuum, then $m(X) = \infty$ or $m(X)$ is an even integer [1]. There exists a planar Peano continuum for which $m(X) = 6$ [2]; in [3], an example D where $m(D) = 4$ is given.

Corollary (3.2) and Theorem (3.1) are each equivalent to the following: X is multicoherent if and only if $m(X) \geq 4$; the construction of $\{A_0, A_1, A_2, A_3\}$ in Corollary (3.2) shows that for a multicoherent space X , $m(X) \geq 4$. (We acknowledge that it is redundant to speak of two *theorems* as equivalent, however, before the present paper, only special cases (e.g. X is compact [5], or X is perfectly normal [4]) were known.)

The example D of [3] (where $m(D) = 4$) shows that Corollary (3.2) (and its equivalent statements) is a best possible result in the following sense: There exists a multicoherent space D such that whenever B_1, B_2 and B_3 are pairwise disjoint simple continua, one of the pairs, $B_1 \cup B_2, B_2 \cup B_3$ or $B_3 \cup B_1$, fails to separate X . (For if there existed a triple B_1, B_2 and B_3 so that all the pairs $B_1 \cup B_2, B_2 \cup B_3$ and $B_3 \cup B_1$ separated X , then $m(D) \geq 6$ [5, Theorem 3]; of course this is not true in general, since $m(D) = 4$.)

REFERENCES

1. Harold Bell, Ezra Brown, R. F. Dickman, Jr. and E. L. Green, *Circularity of graphs and continua: Topology*, Fund. Math. **112** (1981), 103–110.
2. Harold Bell and R. F. Dickman, Jr., *A counterexample to a conjecture of A. H. Stone*, Proc. Amer. Math. Soc. **71** (1978), 305–308.

3. _____, *The end of a conjecture of A. H. Stone*, *Topology Appl.* (submitted).
4. R. F. Dickman, Jr., *A proof of a conjecture of A. H. Stone*, *Proc. Amer. Math. Soc.* **80** (1980), 177–180.
5. _____, *Multicoherent spaces*, *Fund. Math.* **91** (1976), 219–229.
6. R. F. Dickman, Jr. and L. R. Rubin, *C-separated sets and unicoherence*, *Fund. Math.* **83** (1973), 25–33.
7. J. H. V. Hunt, *The Phragmen-Brouwer theorem for separated sets*, *Bol. Soc. Sci. Mexicana* **19** (1974), 26–35.
8. J. H. V. Hunt and E. D. Tymchatyn, *A theorem on involutions on unicoherent spaces*, *Quart. J. Math. Oxford Ser. (2)* **32** (1981), 57–67.
9. Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, Van Nostrand Rheinhold, Princeton, N.J., 1960.
10. A. H. Stone, *Incidence relations in unicoherent spaces*, *Trans. Amer. Math. Soc.* **65** (1949), 427–447.
11. _____, *Incidence relations in multicoherent spaces. I*, *Trans. Amer. Math. Soc.* **66** (1949), 389–406.
12. _____, *Incidence relations in multicoherent spaces. II*, *Canad. J. Math.* **2** (1950), 461–480.
13. _____, *Incidence relations in multicoherent spaces. III*, *Pacific J. Math.* **2** (1952), 99–126.
14. G. T. Whyburn, *Analytic topology*, *Amer. Math. Soc. Colloq. Publ.*, vol. 28, Amer. Math. Soc., Providence, R.I., 1942.

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