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A NEW q-LAGRANGE FORMULA AND SOME APPLICATIONS

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ABSTRACT. A new q-extension of the Lagrange-Bürmann expansion and related formulas are proved. Finally we give a method to find q-generalizations of Riordan's inverse relations.

1. Introduction. The Lagrange-Bürmann formula solves the problem of computing the coefficients c_k in the expansion $g(z) = \sum_{k=0}^{\infty} c_k z^k / f^k(z)$, where $f(z)$ and $g(z)$ are given formal power series (fps) with $f(0) \neq 0$. In this paper we shall use a method introduced by Egorychev [2]. Consider $a(z)$ a Laurent series (Ls), then $\operatorname{coef}_z(a(z)dz)$ denotes the coefficient of z^{-1} in $a(z)$. The two (equivalent) versions of the Lagrange formula can be rewritten as

$$c_n = \operatorname*{coef}_z \left(g(z) f^n(z) \left(1 - z \frac{df(z)/dz}{f(z)} \right) \frac{dz}{z^{n+1}} \right) \tag{1.1}$$

or

$$c_n = \frac{1}{n} \operatorname*{coef}_z \left(\frac{d}{dz} g(z) f^n(z) \frac{dz}{z^n} \right) \quad \text{for } n \geqslant 1. \tag{1.2}$$

Jackson [7] and Carlitz [1] found q-analogues in special cases connected with Abel- and Laguerre polynomials, respectively. Garsia and Joni [3, 4] gave a very nice q-extension of (1.1), but it did not contain Jackson's special case. A q-extension containing both Jackson's and Carlitz's results is due to Hofbauer [6]. His results are special cases of Theorem 1 in this paper.

2. Definitions. Let q be a fixed real number with $q \neq 0, 1$. Then we define, as usual, $[\alpha] = (q^\alpha - 1)/(q - 1)$, $[n]! = [n] \cdot [n-1] \cdots [1]$, $[0]! = 1$ and $\left[\begin{smallmatrix}\alpha\\ n\end{smallmatrix}\right] = [\alpha][\alpha - 1] \cdots [\alpha - n + 1]/[n]!$. We introduce the q-difference operator D_q by

$$D_q f(z) = (f(qz) - f(z))/(q-1)z. \tag{2.1}$$

Since $D_q z^n = [n] z^{n-1}$, D_q is a linear operator on the set of Ls. If $a(z)$ is an Ls, the following property holds:

$$\operatorname*{coef}_z \left(D_q a(z)\, dz \right) = 0. \tag{2.2}$$

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The q-exponential function $e_q(z) = \sum_{k=0}^{\infty} z^k/[k]!$ satisfies the differential equation $D_q e_q(z) = e_q(z)$, which is equivalent to

$$e_q(qz) = (1 + (q-1)z)e_q(z). \tag{2.3}$$

Finally, we define

$$p_\alpha(1, z) = \frac{e_q(q^\alpha z/(1-q))}{e_q(z/(1-q))}. \tag{2.4}$$

Use of (2.1) and (2.3) gives $D_q p_\alpha(1, z) = -[\alpha]p_{\alpha-1}(1, qz)$ and by iteration $D_q^k p_\alpha(1, z) = (-1)^k q^{\binom{k}{2}} p_{\alpha-k}(1, q^k z)[\alpha][\alpha-1] \cdots [\alpha-k+1]$. Therefore, we have

$$p_\alpha(1, z) = \sum_{k=0}^{\infty} \frac{D_q^k p_\alpha(1, t)\big|_{t=0}}{[k]!} z^k = \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} \alpha \\ k \end{bmatrix} z^k.$$

3. The Lagrange formula. Hofbauer's idea is based on the observation that

$$\frac{d}{dz} f^n(z) = n \cdot \frac{f'(z)}{f(z)} \cdot f^n(z).$$

This leads to

DEFINITION 1. The fps $\varphi_\alpha(z)$, $\alpha \in \mathbf{R}$ (= real numbers), are called q-powers, if there is a fixed fps $\varphi(z)$ such that $\varphi_\alpha(0) \neq 0$ for all α and

$$D_q \varphi_\alpha(z) = [\alpha]\varphi(z)\varphi_\alpha(z). \tag{3.1}$$

EXAMPLE 1. Let us suppose $a, b \in \mathbf{R}$, and m is a positive integer, then $e_{q^m}((a[\alpha]+b)z^m)/e_{q^m}(bz^m)$ are q-powers corresponding to

$$\varphi(z) = \frac{a[m]z^{m-1}}{1 + (q^m - 1)bz^m}.$$

To see this we only have to use (2.1) and (2.3), which leads to

$$D_q\left(\frac{e_{q^m}((a[\alpha]+b)z^m)}{e_{q^m}(bz^m)}\right) = \frac{a[\alpha][m]z^{m-1}}{(1 + (q^m - 1)bz^m)} \frac{e_{q^m}((a[\alpha]+b)z^m)}{e_{q^m}(bz^m)}.$$

LEMMA 1. *Let $\varphi_\alpha(z)$ and $\phi_\alpha(z)$ be q-powers corresponding to $\varphi(z)$ and $\phi(z)$, respectively. Take λ, $\mu \in \mathbf{R}$, then*

$$\operatorname*{coef}_z \left(\frac{\varphi_{n+\lambda}(z)/\phi_{n-\mu}(qz)}{\varphi_{k+\lambda}(qz)/\phi_{-k-\mu}(z)} \cdot \frac{(1 - z\varphi(z) - z\phi(z) + z^2\varphi(z)\phi(z)(1 - q^{\lambda-\mu}))}{z^{n-k+1}} dz \right) = \delta_{nk} \tag{3.2}$$

where δ_{nk} is the Kronecker delta.

PROOF. Observe that (3.1) is equivalent to

$$\varphi_\alpha(qz) = (1 + (q^\alpha - 1)z\varphi(z))\varphi_\alpha(z) \tag{3.3}$$

and ϕ_α the same. By using (2.1) and (3.3) we get

$$D_q\left(\frac{\varphi_{n+\lambda}(z)/\phi_{-n-\mu}(z)}{\varphi_{k+\lambda}(z)/\phi_{-k-\mu}(z)\cdot z^{n-k}}\right)=\frac{\varphi_{n+\lambda}(z)/\phi_{-n-\mu}(qz)}{\varphi_{k+\lambda}(qz)/\phi_{-k-\mu}(z)}\frac{1}{(q-1)z^{n-k+1}q^{n-k}}$$
$$\cdot\Big[(1+(q^{n+\lambda}-1)z\varphi(z))(1+(q^{-k-\mu}-1)z\phi(z))$$
$$-q^{n-k}(1+(q^{-n-\mu}-1)z\phi(z))(1+(q^{k+\lambda}-1)z\varphi(z))\Big]$$
$$=-\frac{[n-k]}{q^{n-k}}\frac{\varphi_{n+\lambda}(z)/\phi_{-n-\mu}(qz)}{\varphi_{k+\lambda}(qz)/\phi_{-k-\mu}(z)}$$
$$\cdot\frac{(1-z\varphi(z)-z\phi(z)+z^2\varphi(z)\phi(z)(1-q^{\lambda-\mu}))}{z^{n-k+1}}.$$

If $n\neq k$ we have proved (3.2) by remembering (2.2). The case $n=k$ can be evaluated directly. □

We now obtain the q-extensions of (1.1) and (1.2) as easy consequences of this lemma.

THEOREM 1. *With the assumptions of Lemma* 1 *and* $g(z)$ *an fps we have*:
(A) *If*

$$g(z)=\sum_{k=0}^{\infty}c_k\frac{z^k}{\varphi_{k+\lambda}(qz)/\phi_{-k-\mu}(z)},$$

then

$$c_n=\operatorname*{coef}_z\left(g(z)\frac{\varphi_{n+\lambda}(z)}{\phi_{-n-\mu(qz)}}(1-z\varphi(z)-z\phi(z)+z^2\varphi(z)\phi(z)(1-q^{\lambda-\mu}))\frac{dz}{z^{n+1}}\right).$$

(B) *If*

$$g(z)=g(0)+\sum_{k=1}^{\infty}c_k\frac{z^k}{\varphi_k(z)/\phi_{-k}(z)},$$

then

$$c_n=\frac{1}{[n]}\operatorname*{coef}_z\left(D_q(g(z))\frac{\varphi_n(z)}{\phi_{-n}(qz)}\frac{dz}{z^n}\right).$$

PROOF. (A) is obvious. Concerning (B), evaluate

$$D_q\left(\frac{z^k}{\varphi_k(z)/\phi_{-k}(z)}\right)=\frac{z^k}{\varphi_k(qz)/\phi_{-k}(z)}\frac{1}{(q-1)z}$$
$$\cdot\left(q^k(1+(q^{-k}-1)z\phi(z))-(1+(q^k-1)z\varphi(z))\right)$$
$$=[k]\frac{z^{k-1}}{\varphi_k(qz)/\phi_{-k}(z)}(1-z\varphi(z)-z\phi(z)).$$

Therefore,

$$\operatorname*{coef}_z\left(D_q\left(g(z)\frac{\varphi_n(z)}{\phi_{-n}(qz)}\ \frac{dz}{z^n}\right)\right.$$
$$=\operatorname*{coef}_z\left(\sum_{k=1}^{\infty}[k]c_k\frac{\varphi_n(z)/\phi_{-n}(qz)}{\varphi_k(qz)/\phi_{-k}(z)}\ \frac{(1-z\varphi(z)-z\phi(z))}{z^{n-k+1}}dz\right)=[n]c_n$$

by Lemma 1, setting $\lambda=\mu=0$. □

4. Examples.

EXAMPLE 2 (JACKSON'S SPECIAL CASE). By setting $b=0$ and $m=1$ in Example 1, we see that $e_q(a[\alpha]z)$ are q-powers corresponding to a. Use of Theorem 1 gives: If

$$g(z)=\sum_{k=0}^{\infty}c_k\frac{z^k}{e_q(aq[k+\lambda]z)},$$

then

$$c_n=\operatorname*{coef}_z\left(g(z)e_q(a[n+\lambda]z)(1-az)\ \frac{dz}{z^{n+1}}\right)$$

and if

$$g(z)=g(0)+\sum_{k=1}^{\infty}c_k\frac{z^k}{e_q(a[k]z)},$$

then

$$c_n=\frac{1}{[n]}\operatorname*{coef}_z\left(D_q(g(z))e_q(a[n]z)\ \frac{dz}{z^n}\right).$$

EXAMPLE 3 (CARLITZ'S SPECIAL CASE). Take $a=-1$, $b=1/(1-q)$ and $m=1$. Because of (2.4) we get: $p_\alpha(1,z)$ are q-powers corresponding to $-1/(1-z)$. Finally, simple calculations show that by Theorem 1 we have: If

$$g(z)=\sum_{k=0}^{\infty}c_k\frac{z^k}{p_{k+\lambda}(1,qz)},$$

then

$$c_n=\operatorname*{coef}_z\left(g(z)p_{n+\lambda-1}(1,qz)\ \frac{dz}{z^{n+1}}\right)$$

and if

$$g(z)=g(0)+\sum_{k=1}^{\infty}c_k\frac{z^k}{p_k(1,z)},$$

then

$$c_n=\frac{1}{[n]}\operatorname*{coef}_z\left(D_q(g(z))p_n(1,z)\ \frac{dz}{z^n}\right).$$

EXAMPLE 4. In Theorem 1, take $\varphi_\alpha(z) = \phi_\alpha(z) = p_\alpha(1, z)$ and $\mu = 0$. Again we avoid the calculations, which lead to: If

$$g(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{p_{2k+\lambda+1}(1, q^{-k}z)}$$

then

$$c_n = \operatorname*{coef}_z \left(g(z) p_{2n+\lambda-1}(1, q^{-n+1}z)(1 - q^\lambda z^2) \frac{dz}{z^{n+1}} \right)$$

and if

$$g(z) = g(0) + \sum_{k=1}^{\infty} c_k \frac{z^k}{p_{2k}(1, q^{-k}z)},$$

then

$$c_n = \frac{1}{[n]} \operatorname*{coef}_z \left(D_q(g(z)) p_{2n-1}(1, q^{-n+1}z)(1 - z) \frac{dz}{z^n} \right).$$

In [5] I. Gessel and D. Stanton obtain these three examples as special cases of a theorem about q-Lagrange inversion. It seems that with the exception of a few examples this theorem cannot be derived by our theory. It would be very interesting to find the connections between these results.

5. A theorem about inverse relations. In [2] Egorychev gave a method to prove all the inverse relations of Riordan [8]. To find q-inverse relations we use a more general result based upon the same idea.

THEOREM 2. *Let* $(g_n(z))_{n=0}^{\infty}$, $(G_k(z))_{k=0}^{\infty}$, $(h_n(z))_{n=0}^{\infty}$ *and* $(H_k(z))_{k=0}^{\infty}$ *be sequences of fps with*

$$\operatorname*{coef}_z \left(\frac{g_n(z)}{G_k(z)} \frac{dz}{z^{n-k+1}} \right) = \operatorname*{coef}_z \left(\frac{h_n(z)}{H_k(z)} \frac{dz}{z^{n-k+1}} \right) = \delta_{nk}. \tag{5.1}$$

If $(\alpha_n)_{n=0}^{\infty}$ *and* $(\beta_n)_{n=0}^{\infty}$ *are sequences of real numbers different from zero and* $f(z)$ *is an fps with* $f(0) \neq 0$, *then* $a_n = \sum_{k=0}^{n} c_{nk} b_k$ *holds with*

$$c_{nk} = \frac{\beta_k}{\alpha_n} \operatorname*{coef}_z \left(f(z) \frac{g_n(z)}{H_k(z)} \frac{dz}{z^{n-k+1}} \right)$$

if and only if $b_n = \sum_{k=0}^{n} d_{nk} a_k$ *with*

$$d_{nk} = \frac{\alpha_k}{\beta_n} \operatorname*{coef}_z \left(f(z)^{-1} \frac{h_n(z)}{G_k(z)} \frac{dz}{z^{n-k+1}} \right).$$

PROOF. It is sufficient to prove only one implication; the other follows by symmetry. We show " $\Rightarrow$ ":

$$\alpha_n a_n = \operatorname*{coef}_z \left(\sum_{k=0}^{\infty} \alpha_k a_k \frac{z^k}{G_k(z)} \frac{g_n(z)}{z^{n+1}} dz \right) \tag{5.2}$$

by (5.1). On the other hand, we have

$$\alpha_n a_n = \alpha_n \sum_{k=0}^{n} c_{nk} b_k = \sum_{k=0}^{n} \alpha_n b_k \cdot \frac{\beta_k}{\alpha_n} \operatorname*{coef}_z \left(f(z) \frac{g_n(z)}{H_k(z)} \frac{dz}{z^{n-k+1}} \right) \tag{5.3}$$
$$= \operatorname*{coef}_z \left(f(z) \sum_{k=0}^{\infty} b_k \beta_k \frac{z^k}{H_k(z)} \frac{g_n(z)}{z^{n+1}} dz \right).$$

Since (5.2) and (5.3) hold for every nonnegative integer n, the following equation is true:

$$f(z) \sum_{k=0}^{\infty} b_k \beta_k \frac{z^k}{H_k(z)} = \sum_{k=0}^{\infty} \alpha_k a_k \frac{z^k}{G_k(z)}. \tag{5.4}$$

Use of (5.1) and (5.4) gives

$$\beta_n b_n = \operatorname*{coef}_z \left(\sum_{k=0}^{\infty} \beta_k b_k \frac{z^k}{H_k(z)} \frac{h_n(z)}{z^{n+1}} dz \right)$$
$$= \operatorname*{coef}_z \left(f(z)^{-1} \sum_{k=0}^{\infty} \alpha_k a_k \frac{z^k}{G_k(z)} \frac{h_n(z)}{z^{n+1}} dz \right)$$
$$= \sum_{k=0}^{n} \alpha_k a_k \operatorname*{coef}_z \left(f(z)^{-1} \frac{h_n(z)}{G_k(z)} \frac{dz}{z^{n-k+1}} \right).$$

Note that the last step essentially needs the condition $f(0) \neq 0$. Division by β_n completes the proof. □

Obviously, Lemma 1 gives many examples for the pairs $g_n(z)$, $G_k(z)$ and $h_n(z)$, $H_k(z)$ by using Example 1. Indeed, it is possible to find explicit q-analogues of Riordan's inverse relations to Chebyshev-, Legendre- or Abel-type. Some simple examples are listed below.

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+p \\ n-k \end{bmatrix} b_k \Leftrightarrow b_n = \sum_{k=0}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n+p \\ n-k \end{bmatrix} a_k,$$
$$a_n = \sum_{k=0}^{[n/2]} (-1)^{k} \begin{bmatrix} n+p \\ k \end{bmatrix} b_{n-2k} \Leftrightarrow b_n = \sum_{k=0}^{[n/2]} q^{\binom{k}{2}} \begin{bmatrix} n+p-k \\ k \end{bmatrix} \frac{[n+p]}{[n+p-k]} a_{n-2k},$$
$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} 2n+p \\ n-k \end{bmatrix} b_k \Leftrightarrow b_n = \sum_{k=0}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n+p+k \\ n-k \end{bmatrix} \frac{[2n+p]}{[n+p+k]} a_k,$$
$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} [n+p]^{n-k} b_k \Leftrightarrow b_n = \sum_{k=0}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} [n+p][k+p]^{n-k-1} a_k.$$

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