

A NEW q -LAGRANGE FORMULA AND SOME APPLICATIONS

CHRISTIAN KRATTENTHALER

ABSTRACT. A new q -extension of the Lagrange-Bürmann expansion and related formulas are proved. Finally we give a method to find q -generalizations of Riordan's inverse relations.

1. Introduction. The Lagrange-Bürmann formula solves the problem of computing the coefficients c_k in the expansion $g(z) = \sum_{k=0}^{\infty} c_k z^k / f^k(z)$, where $f(z)$ and $g(z)$ are given formal power series (fps) with $f(0) \neq 0$. In this paper we shall use a method introduced by Egorychev [2]. Consider $a(z)$ a Laurent series (Ls), then $\text{coef}_z(a(z) dz)$ denotes the coefficient of z^{-1} in $a(z)$. The two (equivalent) versions of the Lagrange formula can be rewritten as

$$(1.1) \quad c_n = \text{coef}_z \left(g(z) f^n(z) \left(1 - z \frac{df(z)/dz}{f(z)} \right) \frac{dz}{z^{n+1}} \right)$$

or

$$(1.2) \quad c_n = \frac{1}{n} \text{coef}_z \left(\frac{d}{dz} g(z) f^n(z) \frac{dz}{z^n} \right) \quad \text{for } n \geq 1.$$

Jackson [7] and Carlitz [1] found q -analogues in special cases connected with Abel- and Laguerre polynomials, respectively. Garsia and Joni [3, 4] gave a very nice q -extension of (1.1), but it did not contain Jackson's special case. A q -extension containing both Jackson's and Carlitz's results is due to Hofbauer [6]. His results are special cases of Theorem 1 in this paper.

2. Definitions. Let q be a fixed real number with $q \neq 0, 1$. Then we define, as usual, $[\alpha] = (q^\alpha - 1)/(q - 1)$, $[n]! = [n] \cdot [n - 1] \cdots [1]$, $[0]! = 1$ and $[\alpha_n] = [\alpha][\alpha - 1] \cdots [\alpha - n + 1]/[n]!$. We introduce the q -difference operator D_q by

$$(2.1) \quad D_q f(z) = (f(qz) - f(z))/(q - 1)z.$$

Since $D_q z^n = [n]z^{n-1}$, D_q is a linear operator on the set of Ls. If $a(z)$ is an Ls, the following property holds:

$$(2.2) \quad \text{coef}_z (D_q a(z) dz) = 0.$$

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The q -exponential function $e_q(z) = \sum_{k=0}^{\infty} z^k / [k]!$ satisfies the differential equation $D_q e_q(z) = e_q(z)$, which is equivalent to

$$(2.3) \quad e_q(qz) = (1 + (q - 1)z)e_q(z).$$

Finally, we define

$$(2.4) \quad p_{\alpha}(1, z) = \frac{e_q(q^{\alpha}z / (1 - q))}{e_q(z / (1 - q))}.$$

Use of (2.1) and (2.3) gives $D_q p_{\alpha}(1, z) = -[\alpha] p_{\alpha-1}(1, qz)$ and by iteration $D_q^k p_{\alpha}(1, z) = (-1)^k q^{\binom{k}{2}} p_{\alpha-k}(1, q^k z) [\alpha][\alpha - 1] \cdots [\alpha - k + 1]$. Therefore, we have

$$p_{\alpha}(1, z) = \sum_{k=0}^{\infty} \frac{D_q^k p_{\alpha}(1, t) |_{t=0}}{[k]!} z^k = \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} [\alpha]_k z^k.$$

3. The Lagrange formula. Hofbauer's idea is based on the observation that

$$\frac{d}{dz} f^n(z) = n \cdot \frac{f'(z)}{f(z)} \cdot f^n(z).$$

This leads to

DEFINITION 1. The fps $\varphi_{\alpha}(z)$, $\alpha \in \mathbf{R}$ (= real numbers), are called q -powers, if there is a fixed fps $\varphi(z)$ such that $\varphi_{\alpha}(0) \neq 0$ for all α and

$$(3.1) \quad D_q \varphi_{\alpha}(z) = [\alpha] \varphi(z) \varphi_{\alpha}(z).$$

EXAMPLE 1. Let us suppose $a, b \in \mathbf{R}$, and m is a positive integer, then $e_{q^m}((a[\alpha] + b)z^m) / e_{q^m}(bz^m)$ are q -powers corresponding to

$$\varphi(z) = \frac{a[m]z^{m-1}}{1 + (q^m - 1)bz^m}.$$

To see this we only have to use (2.1) and (2.3), which leads to

$$D_q \left(\frac{e_{q^m}((a[\alpha] + b)z^m)}{e_{q^m}(bz^m)} \right) = \frac{a[\alpha][m]z^{m-1}}{(1 + (q^m - 1)bz^m)} \frac{e_{q^m}((a[\alpha] + b)z^m)}{e_{q^m}(bz^m)}.$$

LEMMA 1. Let $\varphi_{\alpha}(z)$ and $\phi_{\alpha}(z)$ be q -powers corresponding to $\varphi(z)$ and $\phi(z)$, respectively. Take $\lambda, \mu \in \mathbf{R}$, then

$$(3.2) \quad \text{coef}_z \left(\frac{\varphi_{n+\lambda}(z) / \phi_{n-\mu}(qz)}{\varphi_{k+\lambda}(qz) / \phi_{-k-\mu}(z)} \cdot \frac{(1 - z\varphi(z) - z\phi(z) + z^2\varphi(z)\phi(z)(1 - q^{\lambda-\mu}))}{z^{n-k+1}} dz \right) = \delta_{nk}$$

where δ_{nk} is the Kronecker delta.

PROOF. Observe that (3.1) is equivalent to

$$(3.3) \quad \varphi_{\alpha}(qz) = (1 + (q^{\alpha} - 1)z\varphi(z))\varphi_{\alpha}(z)$$

and ϕ_α the same. By using (2.1) and (3.3) we get

$$\begin{aligned} D_q \left(\frac{\varphi_{n+\lambda}(z)/\phi_{-n-\mu}(z)}{\varphi_{k+\lambda}(z)/\phi_{-k-\mu}(z) \cdot z^{n-k}} \right) &= \frac{\varphi_{n+\lambda}(z)/\phi_{-n-\mu}(qz)}{\varphi_{k+\lambda}(qz)/\phi_{-k-\mu}(z)} \frac{1}{(q-1)z^{n-k+1}q^{n-k}} \\ &\quad \cdot \left[(1+(q^{n+\lambda}-1)z\varphi(z))(1+(q^{-k-\mu}-1)z\phi(z)) \right. \\ &\quad \left. - q^{n-k}(1+(q^{-n-\mu}-1)z\phi(z))(1+(q^{k+\lambda}-1)z\varphi(z)) \right] \\ &= - \frac{[n-k]}{q^{n-k}} \frac{\varphi_{n+\lambda}(z)/\phi_{-n-\mu}(qz)}{\varphi_{k+\lambda}(qz)/\phi_{-k-\mu}(z)} \\ &\quad \cdot \frac{(1-z\varphi(z)-z\phi(z)+z^2\varphi(z)\phi(z)(1-q^{\lambda-\mu}))}{z^{n-k+1}}. \end{aligned}$$

If $n \neq k$ we have proved (3.2) by remembering (2.2). The case $n = k$ can be evaluated directly. \square

We now obtain the q -extensions of (1.1) and (1.2) as easy consequences of this lemma.

THEOREM 1. *With the assumptions of Lemma 1 and $g(z)$ an fps we have:*

(A) *If*

$$g(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{\varphi_{k+\lambda}(qz)/\phi_{-k-\mu}(z)},$$

then

$$c_n = \operatorname{coef}_z \left(g(z) \frac{\varphi_{n+\lambda}(z)}{\phi_{-n-\mu}(qz)} (1-z\varphi(z)-z\phi(z)+z^2\varphi(z)\phi(z)(1-q^{\lambda-\mu})) \frac{dz}{z^{n+1}} \right).$$

(B) *If*

$$g(z) = g(0) + \sum_{k=1}^{\infty} c_k \frac{z^k}{\varphi_k(z)/\phi_{-k}(z)},$$

then

$$c_n = \frac{1}{[n]} \operatorname{coef}_z \left(D_q(g(z)) \frac{\varphi_n(z)}{\phi_{-n}(qz)} \frac{dz}{z^n} \right).$$

PROOF. (A) is obvious. Concerning (B), evaluate

$$\begin{aligned} D_q \left(\frac{z^k}{\varphi_k(z)/\phi_{-k}(z)} \right) &= \frac{z^k}{\varphi_k(qz)/\phi_{-k}(z)} \frac{1}{(q-1)z} \\ &\quad \cdot (q^k(1+(q^{-k}-1)z\phi(z)) - (1+(q^k-1)z\varphi(z))) \\ &= [k] \frac{z^{k-1}}{\varphi_k(qz)/\phi_{-k}(z)} (1-z\varphi(z)-z\phi(z)). \end{aligned}$$

Therefore,

$$\begin{aligned} & \operatorname{coef}_z \left(D_q \left(g(z) \frac{\varphi_n(z)}{\phi_{-n}(qz)} \frac{dz}{z^n} \right) \right) \\ &= \operatorname{coef}_z \left(\sum_{k=1}^{\infty} [k] c_k \frac{\varphi_n(z)/\phi_{-n}(qz)}{\varphi_k(qz)/\phi_{-k}(z)} \frac{(1 - z\varphi(z) - z\phi(z))}{z^{n-k+1}} dz \right) = [n] c_n \end{aligned}$$

by Lemma 1, setting $\lambda = \mu = 0$. \square

4. Examples.

EXAMPLE 2 (JACKSON'S SPECIAL CASE). By setting $b = 0$ and $m = 1$ in Example 1, we see that $e_q(a[\alpha]z)$ are q -powers corresponding to a . Use of Theorem 1 gives: If

$$g(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{e_q(aq[k + \lambda]z)},$$

then

$$c_n = \operatorname{coef}_z \left(g(z) e_q(a[n + \lambda]z) (1 - az) \frac{dz}{z^{n+1}} \right)$$

and if

$$g(z) = g(0) + \sum_{k=1}^{\infty} c_k \frac{z^k}{e_q(a[k]z)},$$

then

$$c_n = \frac{1}{[n]} \operatorname{coef}_z \left(D_q(g(z)) e_q(a[n]z) \frac{dz}{z^n} \right).$$

EXAMPLE 3 (CARLITZ'S SPECIAL CASE). Take $a = -1$, $b = 1/(1 - q)$ and $m = 1$. Because of (2.4) we get: $p_\alpha(1, z)$ are q -powers corresponding to $-1/(1 - z)$. Finally, simple calculations show that by Theorem 1 we have: If

$$g(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{p_{k+\lambda}(1, qz)},$$

then

$$c_n = \operatorname{coef}_z \left(g(z) p_{n+\lambda-1}(1, qz) \frac{dz}{z^{n+1}} \right)$$

and if

$$g(z) = g(0) + \sum_{k=1}^{\infty} c_k \frac{z^k}{p_k(1, z)},$$

then

$$c_n = \frac{1}{[n]} \operatorname{coef}_z \left(D_q(g(z)) p_n(1, z) \frac{dz}{z^n} \right).$$

EXAMPLE 4. In Theorem 1, take $\varphi_\alpha(z) = \phi_\alpha(z) = p_\alpha(1, z)$ and $\mu = 0$. Again we avoid the calculations, which lead to: If

$$g(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{p_{2k+\lambda+1}(1, q^{-k}z)}$$

then

$$c_n = \operatorname{coef}_z \left(g(z) p_{2n+\lambda-1}(1, q^{-n+1}z) (1 - q^\lambda z^2) \frac{dz}{z^{n+1}} \right)$$

and if

$$g(z) = g(0) + \sum_{k=1}^{\infty} c_k \frac{z^k}{p_{2k}(1, q^{-k}z)},$$

then

$$c_n = \frac{1}{[n]} \operatorname{coef}_z \left(D_q(g(z)) p_{2n-1}(1, q^{-n+1}z) (1 - z) \frac{dz}{z^n} \right).$$

In [5] I. Gessel and D. Stanton obtain these three examples as special cases of a theorem about q -Lagrange inversion. It seems that with the exception of a few examples this theorem cannot be derived by our theory. It would be very interesting to find the connections between these results.

5. A theorem about inverse relations. In [2] Egorychev gave a method to prove all the inverse relations of Riordan [8]. To find q -inverse relations we use a more general result based upon the same idea.

THEOREM 2. Let $(g_n(z))_{n=0}^\infty$, $(G_k(z))_{k=0}^\infty$, $(h_n(z))_{n=0}^\infty$ and $(H_k(z))_{k=0}^\infty$ be sequences of fps with

$$(5.1) \quad \operatorname{coef}_z \left(\frac{g_n(z)}{G_k(z)} \frac{dz}{z^{n-k+1}} \right) = \operatorname{coef}_z \left(\frac{h_n(z)}{H_k(z)} \frac{dz}{z^{n-k+1}} \right) = \delta_{nk}.$$

If $(\alpha_n)_{n=0}^\infty$ and $(\beta_n)_{n=0}^\infty$ are sequences of real numbers different from zero and $f(z)$ is an fps with $f(0) \neq 0$, then $a_n = \sum_{k=0}^n c_{nk} b_k$ holds with

$$c_{nk} = \frac{\beta_k}{\alpha_n} \operatorname{coef}_z \left(f(z) \frac{g_n(z)}{H_k(z)} \frac{dz}{z^{n-k+1}} \right)$$

if and only if $b_n = \sum_{k=0}^n d_{nk} a_k$ with

$$d_{nk} = \frac{\alpha_k}{\beta_n} \operatorname{coef}_z \left(f(z)^{-1} \frac{h_n(z)}{G_k(z)} \frac{dz}{z^{n-k+1}} \right).$$

PROOF. It is sufficient to prove only one implication; the other follows by symmetry. We show “ \Rightarrow ”:

$$(5.2) \quad \alpha_n a_n = \operatorname{coef}_z \left(\sum_{k=0}^{\infty} \alpha_k a_k \frac{z^k}{G_k(z)} \frac{g_n(z)}{z^{n+1}} dz \right)$$

by (5.1). On the other hand, we have

$$(5.3) \quad \alpha_n a_n = \alpha_n \sum_{k=0}^n c_{nk} b_k = \sum_{k=0}^n \alpha_n b_k \cdot \frac{\beta_k}{\alpha_n} \operatorname{coef}_z \left(f(z) \frac{g_n(z)}{H_k(z)} \frac{dz}{z^{n-k+1}} \right) \\ = \operatorname{coef}_z \left(f(z) \sum_{k=0}^{\infty} b_k \beta_k \frac{z^k}{H_k(z)} \frac{g_n(z)}{z^{n+1}} dz \right).$$

Since (5.2) and (5.3) hold for every nonnegative integer n , the following equation is true:

$$(5.4) \quad f(z) \sum_{k=0}^{\infty} b_k \beta_k \frac{z^k}{H_k(z)} = \sum_{k=0}^{\infty} \alpha_k a_k \frac{z^k}{G_k(z)}.$$

Use of (5.1) and (5.4) gives

$$\beta_n b_n = \operatorname{coef}_z \left(\sum_{k=0}^{\infty} \beta_k b_k \frac{z^k}{H_k(z)} \frac{h_n(z)}{z^{n+1}} dz \right) \\ = \operatorname{coef}_z \left(f(z)^{-1} \sum_{k=0}^{\infty} \alpha_k a_k \frac{z^k}{G_k(z)} \frac{h_n(z)}{z^{n+1}} dz \right) \\ = \sum_{k=0}^n \alpha_k a_k \operatorname{coef}_z \left(f(z)^{-1} \frac{h_n(z)}{G_k(z)} \frac{dz}{z^{n-k+1}} \right).$$

Note that the last step essentially needs the condition $f(0) \neq 0$. Division by β_n completes the proof. \square

Obviously, Lemma 1 gives many examples for the pairs $g_n(z)$, $G_k(z)$ and $h_n(z)$, $H_k(z)$ by using Example 1. Indeed, it is possible to find explicit q -analogues of Riordan's inverse relations to Chebyshev-, Legendre- or Abel-type. Some simple examples are listed below.

$$a_n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n+p \\ n-k \end{bmatrix} b_k \Leftrightarrow b_n = \sum_{k=0}^n q^{\binom{n-k}{2}} \begin{bmatrix} n+p \\ n-k \end{bmatrix} a_k, \\ a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n+p \\ k \end{bmatrix} b_{n-2k} \Leftrightarrow b_n = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n+p-k \\ k \end{bmatrix} \frac{[n+p]}{[n+p-k]} a_{n-2k}, \\ a_n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} 2n+p \\ n-k \end{bmatrix} b_k \Leftrightarrow b_n = \sum_{k=0}^n q^{\binom{n-k}{2}} \begin{bmatrix} n+p+k \\ n-k \end{bmatrix} \frac{[2n+p]}{[n+p+k]} a_k, \\ a_n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} [n+p]^{n-k} b_k \Leftrightarrow b_n = \sum_{k=0}^n q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} [n+p][k+p]^{n-k-1} a_k.$$

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INSTITUT FÜR MATHEMATIK, A-1090 WIEN, STRUDLHOFGASSE 4, AUSTRIA