ON THE SEMISIMPLECTICITY OF SKEW POLYNOMIAL RINGS

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Abstract. Let $R$ be a ring satisfying the maximal condition on annihilator left ideals and $\sigma$ be an automorphism of $R$. We show that the Jacobson radical of the skew polynomial ring $R_\sigma[x]$ is nonzero if and only if the prime radical of $R_\sigma[x]$ is nonzero. Furthermore, it is so if and only if the prime radical of $R$ is nonzero. In general, an example is given of a commutative semisimple algebra $R$ and an automorphism $\sigma$ such that $R_\sigma[x]$ is prime but the Levitzki radical of $R_\sigma[x]$ is nonzero.

0. Introduction. Let $R$ be a ring and $\sigma$ an automorphism of $R$. In this paper we study the semisimplicity of the skew polynomial ring $R_\sigma[x]$, and that of the skew group ring $R_\sigma\langle x \rangle$ of the infinite cyclic group.

If $R$ satisfies the maximal chain condition on annihilator left ideals, then we prove that the conditions: (i) $J(R_\sigma\langle x \rangle) \neq 0$, (ii) $J(R_\sigma[x]) \neq 0$, (iii) $P(R) \neq 0$, (iv) $P(R_\sigma\langle x \rangle) \neq 0$, (v) $P(R_\sigma[x]) \neq 0$, are equivalent (Theorem 2.1). In general, we construct a commutative semisimple algebra $R$ and an algebra automorphism $\sigma$ where we prove, using Van der Waerden's result about arbitrary long arithmetic progressions, that the Levitzki radical $\mathcal{E}(R_\sigma[x])$ of $R_\sigma[x]$ is nonzero but the ring $R_\sigma[x]$ is prime.

Finally, we discuss the semisimplicity question for $R_\sigma[x]$, where $R$ is an arbitrary commutative ring. In the example (Example 3.2) $R$ is a commutative ring and $\mathcal{E}(R_\sigma[x]) \neq 0$, but $P(R_\sigma[x]) = 0$. When $R$ is a commutative ring, we raise the question: $J(R_\sigma[x]) \neq 0$ if and only if $\mathcal{E}(R_\sigma[x]) \neq 0$. We have settled this question in the affirmative in some cases.

1. Definitions and preliminaries.

1.1. Let $R$ be a ring with 1 and $\sigma$ be an automorphism of $R$. Then by the skew polynomial ring $R_\sigma[x]$, we mean the ring

$$R_\sigma[x] = \left\{ \sum_{i \geq 0} r_i x^i : r_i \in R \text{ and almost all } r_i \text{ are zero} \right\}$$

with addition componentwise and multiplication defined by the rule $x r = \sigma(r)x$. By the skew group ring $R_\sigma\langle x \rangle$, we mean the ring

$$R_\sigma\langle x \rangle = \left\{ \sum_{i \in \mathbb{Z}} r_i x^i : r_i \in R \text{ and almost all } r_i \text{ are zero} \right\}$$
with addition componentwise and multiplication defined by $x'r = \sigma'(r)x'$.

1.2. An element $r \in R$ is said to be $\sigma$-nilpotent if for all natural integers $m$ there exists an integer $n = n(m) > 2$, depending on $m$, such that

$$\sigma^n(r) \cdots \sigma^2(r) \sigma(r) = 0.$$ 

It is said to be $\sigma$-nilpotent of bounded index if we can find a common integer $n$ such that the above equality is satisfied for all natural integers $m$, and the least such $n$ is called the $\sigma$-nilpotency index of $r$. An ideal $I$ of $R$ is said to be $\sigma$-nil if $\sigma(I) = I$ and every element of $I$ is $\sigma$-nilpotent.

2. Main results. In Theorem 3.1 of [1] by Bedi and the author, it is proved that

$$J(R_\sigma[x]) = I \cap J(R) + Ix + Ix^2 + \cdots + Ix^n + \cdots,$$

where $I = \{r \in R: rx \in J(R_\sigma[x])\}$. Here $\sigma(I) = I$. Further, for all $r \in I$, $rx^n \in J(R_\sigma[x])$, and so $rx^n$ is nilpotent by [1, Lemma 2.4]. This proves that $I$ is $\sigma$-nil ideal of $R$. Hence if $J(R_\sigma[x]) \neq 0$ then there exists a nonzero $\sigma$-nil ideal. We will use this result without any further mention.

**Theorem 2.1.** Let $R$ be a ring satisfying the maximal condition on annihilator left ideals and $\sigma$ be an automorphism of $R$. Then the following are equivalent:

1. $J(R_\sigma(x))$ is nonzero.
2. $J(R_\sigma[x])$ is nonzero.
3. $R$ has a nonzero $\sigma$-nil ideal.
4. $R$ has a nonzero right nil ideal.
5. $P(R)$ is nonzero.
6. $P(R_\sigma[x])$ is nonzero.
7. $P(R_\sigma[x])$ is nonzero.

**Proof.** To prove the theorem we first show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1), and then we show that (6) $\Rightarrow$ (7) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2). This result follows from [1, Theorem 3.1].

(2) $\Rightarrow$ (3). If $J(R_\sigma[x])$ then the ring $R$ has a nonzero $\sigma$-nil ideal.

(3) $\Rightarrow$ (4). Let $I$ be a nonzero $\sigma$-nil ideal of $R$. Assume $I$ is not nil and choose $r$ in $I$ such that $r$ is not nilpotent. We claim that there exists $n \geq 1$ such that $\sigma^n(r) \neq 0$. Let us suppose that $\sigma^n(r) = 0$ for all natural integers $n$. Define

$$I_m = \sigma^m(r)R + \sigma^{m+1}(r)R + \cdots + \cdots$$

for all natural integers $m$. Clearly

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \subseteq \cdots.$$ 

Thus

$$\text{Ann}_I(I_1) \subseteq \text{Ann}_I(I_2) \subseteq \text{Ann}_I(I_3) \subseteq \cdots \subseteq \cdots.$$ 

Let $\text{Ann}_I(I_1) = \text{Ann}_I(I_{i+1})$. Since $\sigma^n(r) = 0$ for all natural integers $n$, $\sigma'(r) \in \text{Ann}_I(I_{i+1}) = \text{Ann}_I(I_i)$. Thus

$$\sigma'(r)I_i = 0 \Rightarrow \sigma'(r)\sigma'(r) = 0 \Rightarrow r^2 = 0.$$
Thus, there exists $n \geq 1$ such that $\sigma^n(r) \neq 0$. However $\sigma^n(r) \cdot \cdots \cdot \sigma^n(r) = 0$ for some $t > 1$. We assume that $t$ is minimal. Put $s = \sigma^n(r) \cdot \cdots \cdot \sigma^{(t-1)n}(r)$. If the right ideal $sR$ is not nil, $sk$ is not nilpotent for some $k$. Put $r_1 = r$ and $r_2 = sk$. We show that $\text{Ann}_1(r_1) \subseteq \text{Ann}_1(r_2)$. Since $\sigma^n(r) \cdot \cdots \cdot \sigma^n(r) = 0$, $\sigma^{-n}(r) \in \text{Ann}_1(r_2)$. But $\sigma^{-n}(r) \notin \text{Ann}_1(r_1)$. Thus $\text{Ann}_1(r_1) \subseteq \text{Ann}_1(r_2)$. If $r_2R$ is not nil, then arguing as before we get $r_3$ such that $\text{Ann}_1(r_1) \subseteq \text{Ann}_1(r_2) \subseteq \text{Ann}_1(r_3)$. Continuing in this fashion we will get a nonzero nil right ideal.

(4) $\Rightarrow$ (5). This follows from [6, Lemma 4.7].

(5) $\Rightarrow$ (6). Let

$$\mathcal{F} = \{\text{Ann}_1(Rx) | xRx = 0, x \neq 0\}.$$ 

Since $P(R) \neq 0$, $\mathcal{F}$ is nonempty. Let $\text{Ann}_1(Rr)$ be a maximal element of $\mathcal{F}$. We first show that $rR\sigma^n(r) = 0$ for all integers $n$. If it is not true, then $r\sigma^m(ar) \neq 0$ for an integer $m$. Let $s = \sigma^{-m}(r)ar$. Clearly $sRs = 0$ and $\text{Ann}_1(rs) \supseteq \text{Ann}_1(Rr)$. This inclusion is strict because $\sigma^{-m}(r)Rr \neq 0$ and

$$\sigma^{-m}(r)Rs = \sigma^{-m}(r)R\sigma^{-m}(r)ar = 0.$$

Thus $rR\sigma^n(r) = 0$ for all integers $n$.

Now, observe that $rR\sigma^n(x) = 0$. Hence $P(R\sigma^n(x))$ is nonzero.

(6) $\Rightarrow$ (1). This is clear.

(6) $\Rightarrow$ (7). Since $P(R\sigma^n(x)) \neq 0$, let $I$ be a nonzero nilpotent ideal of $R\sigma^n(x)$. Now, $I \cap R_\sigma[x]$ is a nonzero nilpotent ideal of $R_\sigma[x]$. Hence $P(R_\sigma[x]) \neq 0$.

(7) $\Rightarrow$ (2). This is clear.

3. An example. In this section, we give an example of a commutative semisimple algebra $R$ and an automorphism $\sigma$ such that $R_\sigma[x]$ is prime but the Levitzki radical of $R_\sigma[x]$ is nonzero. In the construction of the example we use the result:

Let $G(k, m)$ denote the least integer such that if $g \geq G(k, m)$ and if $A = \{a_n\}_{n=0}^{g-1}$ is a strictly increasing sequence of integers with bounded gaps $a_n - a_{n-1} \leq m$, $1 \leq n \leq g - 1$, then $A$ contains a $k$-term arithmetic progression. The number $G(k, m)$ does exist [5]. The existence of $G(k, m)$ is an easy consequence of Van der Waerden’s theorem [3, 7].

**Theorem 3.1.** Let $R$ be a commutative ring and $\sigma$ be an automorphism of $R$. If $r \in R$ is a nonzero $\sigma$-nilpotent element of bounded index then $rxR_\sigma[x]$ is a locally nilpotent right ideal of $R_\sigma[x]$.

**Proof.** Let $I = rxR_\sigma[x]$. Clearly $I$ is a right ideal of $R_\sigma[x]$. To prove that $I$ is locally nilpotent, it suffices to prove that the subring $S$ generated by

$$T = \{rs_1x^{i_1}, rs_2x^{i_2}, \ldots, rs_nx^{i_n}\}$$

is nilpotent for all $s_j \in R, i_j \geq 1$. Let $m = \max(i_1, i_2, \ldots, i_n)$ and $k$ be the $\sigma$-nilpotency index of $r$. Let $l = G(k, m)$. We claim that $S' = 0$. For this, it suffices to prove that the product of any $l$ monomials, each of the type given in $T$, is zero. Any typical
such product $P$ is given by

\[
P = \left( r_{s_1}x^{t_1} \right) \left( r_{s_2}x^{t_2} \right) \cdots \left( r_{s_t}x^{t_t} \right)
\]

Thus, it follows that $P = 0$, since the set

\[
\{0, i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_{t_1}\}
\]

contains $k$ numbers in arithmetic progression, and $k$ is the $\sigma$-nilpotency index of $r$.

Here we raise a question: Let $R$ be a commutative ring. Is $J(R_{\sigma}[x])$ nonzero if and only if $J(R_{\sigma}[x])$ is nonzero? If $J(R_{\sigma}[x])$ is nonzero then $R$ has a nonzero $\sigma$-nil ideal, say $I$, and hence Theorem 3.1 settles this question in the affirmative, in the case when $I$ contains a $\sigma$-nilpotent element of bounded index.

**Example 3.2.** Let $K$ be an infinite field and $S$ be a polynomial ring over $K$ in the commuting indeterminates $x_i$ indexed by the set of integers $\mathbb{Z}$. Let $\sigma$ be a map from $S$ to $S$ such that

\[
\sigma(x_i) = x_{i+1} \quad \text{and} \quad \sigma(k) = k \quad \text{for all } k \in \mathbb{Z}.
\]

Then $\sigma$ is an automorphism of $S$. Consider the ideal $I$ of $S$ generated by the set

\[T = \{x_kx_{k+a}x_{k+2a} : k, a \in \mathbb{Z} \text{ and } a \neq 0\}.
\]

Since $\sigma$ permutes the elements of $T$, $\sigma(I) = I$. Let $R = S/I$. Also, $\sigma$ induces an automorphism of $R$ and we denote this automorphism again by $\sigma$. We can think of $R$ as a $k$-linear span of the monomials

\[
\left\{ x_{i_1}^j x_{i_2}^{j_2} \cdots x_{i_k}^{j_k} : \text{no three elements of } i_1, i_2, \ldots, i_k \text{ are in arithmetic progression} \right\}
\]

with usual addition, and we multiply them by using the rule $x_kx_{k+a}x_{k+2a} = 0$ for all $k, a \in \mathbb{Z}$ with $a \neq 0$.

Now, we prove the following: (i) $R$ is semisimple; (ii) $R_{\sigma}[x]$ is prime; (iii) $\mathcal{E}(R_{\sigma}[x])$ is nonzero.

(i) $R$ is semisimple. First, we prove that if $J(R) \neq 0$ then $J(R)$ contains a monomial. If $0 \neq r \in J(R)$ then write $r$ as a polynomial in one of the indeterminates, say $x_{n_1}$; let

\[
r = r_0 + r_1x_{n_1} + \cdots + r_{i_1}x_{n_1}^{i_1}.
\]

Let $\alpha_1, \alpha_2, \ldots, \alpha_{i_1+1}$ be distinct nonzero elements of $K$ and consider the algebra automorphisms $\theta_i$ of $R$, given by

\[
\theta_i(x_{n_1}) = \alpha_i x_{n_1} \quad \theta_i(x_j) = x_j \quad \text{if } j \neq n_1.
\]

Now $\theta_i(r) = r_0 + r_1\alpha_1x_{n_1} + \cdots + r_{i_1}\alpha_{i_1}x_{n_1}^{i_1} \in J(R)$ for all $i$. These equations give that $r_{i_1}x_{n_1}^{i_1}$ is a $K$-linear combination of $\theta_1(r), \theta_2(r), \ldots, \theta_{i_1+1}(r)$, and thus, $r_{i_1}x_{n_1}^{i_1} \in J(R)$. Again, we write the element $r_{i_1}x_{n_1}^{i_1}$ as a polynomial in some other indeterminate and repeat the process. Continuing this way, we finally get that $0 \neq s = kx_{n_1}^{i_1} \cdots x_{n_m}^{i_m} \in J(R)$. Now, since $R$ is a graded ring under the obvious grading and $s$ is a monomial in $J(R)$, by [1, Lemma 2.4], $s$ is nilpotent; however, this is not possible because no three elements of $n_1, n_2, \ldots, n_m$ are in arithmetic progression.
(ii) $R_{a}[x]$ is prime. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ with nonzero $a_n$ and $b_m$ such that

$$f(x)R_{a}[x]g(x) = 0 \Rightarrow f(x)x^{j}g(x) = 0 \quad \text{for all } j \geq 0.$$  

Comparing the coefficient of highest degree term we get that

$$a_n\sigma^{n+j}(b_m) = 0 \quad \text{for all } j \geq 0.$$  

Thus, if $R_{a}[x]$ is not prime, there exists $a = a_n$ and $b = b_m$ such that

$$a\sigma^k(b) = 0 \quad \text{for all } k \geq n.$$  

However, this is not possible and this can be shown by selecting a sufficiently large integer $k$, so that the indeterminates involved in $\sigma^k(b)$ are disjoint from those involved in $a$, and the product of any two monomials in $a\sigma^k(b)$ does not contain three indeterminates whose suffixes are in arithmetic progression.

(iii) $\mathcal{E}(R_{a}[x])$ is nonzero. An element $x_1 \in R$ is $a$-nilpotent of bounded index ($a$-nilpotency index of $x_1$ is three) and hence by Theorem 3.1, $\mathcal{E}(R_{a}[x])$ is nonzero.

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