ON A FAMILY OF SPECIAL LINEAR SYSTEMS
ON ALGEBRAIC CURVES1

EDMOND E. GRIFFIN II

Abstract. \( \mathcal{M}_6 \) is a scheme parametrizing pairs \( L \rightarrow C \) of smooth algebraic curves \( C \) of genus 10 together with line bundles \( L \) of degree 6 such that \( H^0(C, L) \geq 3 \). It is shown that one of the irreducible components of this scheme is nonreduced at every point.

Introduction. In [ACGH] a general method is outlined for computing the dimension of the tangent space \( T_{\mu}^{\mathcal{M}_g} \). \( \mathcal{M}_g \) is the scheme parametrizing pairs \( L \rightarrow C \) in which \( C \) is a smooth curve of genus \( g \), \( L \) is a line bundle such that

1. \( \deg_c(L) = d \),
2. \( \dim H^0(C, L) = h^0(C, L) \geq r + 1 \),

and \( l = L \rightarrow C \) is a “generic” point on \( \mathcal{M}_g \). Specifically, \( l \in \mathcal{M}_{d, g}^{r+1} \).

The dimension of this tangent space is determined by the kernels of two naturally defined maps on the cohomology of \( C \) in \( L \) and \( K_C \). These maps, \( \mu_0 \) and \( \mu_1 \), together with their kernels will be computed explicitly in the case \( l \in \mathcal{M}_{6,10}^{2} - \mathcal{M}_{6,10}^{2+1} \) and \( C \) a hyper-elliptic curve. This calculation will then be used to show that \( \mathcal{M}_{6,10}^{2} \) has a nonreduced component.

It will be clear that these computations generalize to higher genus and degree hyper-elliptic curves, but the actual calculations become fairly intractable. For this reason only the case of \( \mathcal{M}_{6,10}^{2} \) will be considered here.

1. We begin with a review of the basic set up and terminology of [ACGH]. For a fixed, smooth curve \( C \) of genus \( g \), define

\[
\text{Pic}^d(C) = \{ L | \deg_c(L) = d \}
\]

\[
\cup
\]

\[
W^d_c(C) = \{ L | \deg_c(L) = d \text{ and } h^0(C, L) \geq r + 1 \}.
\]

It is well known that

\[
T_{\mu}(\text{Pic}^d(C)) \simeq H^1(C, \mathcal{O}_C).
\]
A first order variation of a line bundle $L$ on $C$ is a commuting diagram,

\[
\begin{array}{ccc}
L & \rightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
C & \rightarrow & C \times S_1 \\
\downarrow & & \downarrow \\
S_0 & \rightarrow & S_1
\end{array}
\]

where $S_n = \text{Spec}(\mathbb{C}[t]/(t^{n+1}))$. The set of isomorphism classes of such diagrams can be canonically identified with $H^1(C, \mathcal{O}_C)$.

The obstruction to extending a section $s \in H^0(C, L)$ to a section of $\mathcal{L}$ can be seen to be $\varphi \cdot s \in H^1(C, L)$, where

\[
\varphi \cdot s \in H^1(C, \mathcal{O}_C) \otimes H^0(C, L) \rightarrow H^1(C, L)
\]

is the usual cup product, and $\varphi \in H^1(C, \mathcal{O}_C)$ is the element corresponding to $\mathcal{L}$. Thus, one sees that, at a point $L \in W^d_d(C) - W^d_{r+1}(C)$, a tangent vector $\varphi$ in $H^1(C, \mathcal{O}_C) = T_L(\text{Pic}^d(C))$ is also in $T_L(W^d_d(C))$ if and only if $\varphi: H^0(C, L) \rightarrow H^1(C, L)$ is the zero map. In other words, $\varphi$ is in the kernel of the map

\[
H^1(C, \mathcal{O}_C) \rightarrow \text{Hom}(H^0(C, L), H^1(C, L)).
\]

Dualizing yields the map

\[
\mu_0: H^0(C, L) \otimes H^0(C, KL^{-1}) \rightarrow H^0(C, K)
\]

such that

\[
T_L(W^d_d(C)) = (\text{Image } \mu_0)^{\perp}
\]

in $H^0(C, L) = r + 1$.

Now the Riemann-Roch theorem implies $h^0(C, KL^{-1}) = g - d + r$. So,

\[
\dim(\text{Image } \mu_0)^{\perp} = g - \dim(\text{Image } \mu_0) = g - ((r + 1)(g - d + r) - \dim(\text{ker } \mu_0))
\]

\[
= (g - (r + 1)(g - d + r)) + \dim(\text{ker } \mu_0) = \text{def } \rho + \dim(\text{ker } \mu_0).
\]

The number $\rho$ is called the Brill-Noether number. So, at a generic $L$,

\[
\dim T_L(W^d_d(C)) = \rho + \dim(\text{ker } \mu_0).
\]

Next we allow $C$ to vary and define (with $g$ fixed)

\[
\text{Pic}^d = \{ L \rightarrow C | \deg_c(L) = d, g(C) = g \}
\]

\[
\text{Pic}^d = \{ L \rightarrow C | \deg_c(L) = d, g(C) = g, \text{ and } h^0(C, L) \geq r + 1 \}.
\]

The fact that these are well-defined schemes with good properties is nontrivial, but is studied carefully in [ACGH].

The tangent space $T_L(\text{Pic}^d)$ can be identified with $H^1(C, \Sigma_L)$, where $\Sigma_L$ is the extension of $\mathcal{O}_C$ by $\Theta_C$,

\[
0 \rightarrow \mathcal{O}_C \rightarrow \Sigma_L \rightarrow \Theta_C \rightarrow 0,
\]

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with extension class
\[ \omega = c_1(L) \in H^1(C, \text{Hom}(\Theta, \Theta)) \approx H^1(C, K). \]

This leads (nontrivially) to the definition of \( \mu_1 : \ker(\mu_0) \to H^0(C, K^2) \) given by

\[
\mu_1 \left( \sum_i s_i \otimes r_i \right) = \frac{\partial s_i}{\partial z_a} \otimes r_i \in H^0(C, K^2),
\]

where \( \sum_i s_i \otimes r_i \) is in \( \ker(\mu_0) \).

Again, by studying obstructions, it can be seen that when \( h^0(C, L) = r + 1 \), we have

\[
\dim T_i(\mathbb{C}_{d, g}) = 3g - 3 + \rho + \dim(\ker(\mu_1)).
\]

Thus in order to compute the dimension of \( T_i(\mathbb{C}_{6,10}) \), one must first find \( \ker(\mu_0) \) and then the dimension of \( \ker(\mu_1) \).

2. Begin by fixing \( L = L \to C \in \mathbb{C}_{6,10} - \mathbb{C}_{3,10} \), where \( C \) is a smooth, genus 10, hyper-elliptic curve. A simple investigation of the map associated to the linear system \( |L| \), \( \varphi_L : C \to \mathbb{P}^2 \) (see [Griffin 2]) shows that, in this case, \( |L| \) has two basepoints and is of the form \( |L| = 2g_2 + Q + R \), with \( Q \neq iR \), where \( i : C \to C \) is the involution on \( C \). We may assume \( |L| = |4P| + Q + R \), where \( P \) is a Weierstrass point on \( C \).

In order to compute \( \mu_0 \), choose a basis for \( H^0(C, L) = H^0(C, 4P + Q + R) \), say \( x_0, x_1, x_2, \) where \( \text{ord}_P(x_i) = 2i \). Then a basis for

\[ H^0(C, KL^{-1}) = H^0(C, \mathcal{O}_C(10P + iQ + iR)) \]

is

\[ \{ x_0^2, x_0x_1, x_0x_2, x_1x_2, x_2^2, y_4 \} \quad \text{where} \quad \text{ord}_P(y_4) = 10. \]

On the other hand, \( H^0(C, K) = H^0(C, \mathcal{O}_C(18P)) \) has basis

\( \{ x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1x_2, x_0x_2^2, x_1x_2^2, x_2y_4, z_5, z_6 \} \quad \text{where} \quad \text{ord}_P(z_j) = 2j + 6. \)

From this it is clear that \( \mu_0 \) is neither injective nor surjective (\( \mu_0 \) just "multiplies" \( x_i \otimes x_j \to x_i x_j x_k \)). Its kernel has dimension 10 and basis given in

\[
\text{Table 1}
\]

\begin{align*}
\nu_1 : & x_0 \otimes x_0x_1 - x_1 \otimes x_0^2 \\
\nu_2 : & x_0 \otimes x_0x_2 - x_2 \otimes x_0^2 \\
\nu_3 : & x_0 \otimes x_1x_2 - x_1 \otimes x_0x_2 \\
\nu_4 : & x_0 \otimes x_1x_2 - x_2 \otimes x_0x_1 \\
\nu_5 : & x_0 \otimes x_2^2 - x_2 \otimes x_0x_2 \\
\nu_6 : & x_0 \otimes y_4 - x_2 \otimes x_1x_2 \\
\nu_7 : & x_1 \otimes x_1x_2 - x_2 \otimes x_0x_2 \\
\nu_8 : & x_1 \otimes x_2^2 - x_2 \otimes x_1x_2 \\
\nu_9 : & x_1 \otimes y_4 - x_2 \otimes x_2^2 \\
\nu_{10} : & x_0 \otimes x_0x_2 - x_1 \otimes x_0x_1 \\
\end{align*}
The elements \( v_6, v_7, v_9, v_{10} \) come from relations in \( H^0(C, K) \), namely \( x_0x_2 - x_1^2 = 0 \), \( x_0y_4 - x_1x_2^2 = 0 \) and \( x_1y_4 - x_2^2 = 0 \). The other elements are Koszul relations.

In order to compute \( \text{ker}(\mu_1) \) we will need

\[
\frac{\partial x_2}{\partial z} = \frac{\partial}{\partial z} \left( \frac{x_2}{x_0} \right) = \frac{2x_1x_0(x_2 + x_0(\partial x_2/\partial z))}{x_0^2}
\]

which comes from the relation \( x_0x_2 - x_1^2 \). The \( \alpha \) subscript will be dropped (for obvious reasons) and the dependence will be denoted by \( \{ \} \)'s.

We now give a table of \( w_i = \mu_1(v_i) \).

### Table 2

| \( w_1 \) | \( \{x_0x_1(\partial x_0/\partial z) - x_0^2(\partial x_1/\partial z)\} \) |
| \( w_2 \) | \( \{2x_0x_2(\partial x_0/\partial z) - 2x_0x_1(\partial x_1/\partial z)\} \) |
| \( w_3 \) | \( \{x_1x_2(\partial x_0/\partial z) - x_0x_2(\partial x_1/\partial z)\} \) |
| \( w_4 \) | \( \{2x_1x_2(\partial x_0/\partial z) - 2x_0x_2(\partial x_1/\partial z)\} \) |
| \( w_5 \) | \( \{2x_1^2(\partial x_0/\partial z) - 2x_1x_2(\partial x_1/\partial z)\} \) |
| \( w_6 \) | \( \{2y_4(\partial x_0/\partial z) - 2x_1^2(\partial x_1/\partial z)\} \) |
| \( w_7 \) | \( \{x_2^2(\partial x_0/\partial z) - x_1x_2(\partial x_1/\partial z)\} \) |
| \( w_8 \) | \( \{y_4(\partial x_0/\partial z) - x_2^2(\partial x_1/\partial z)\} \) |
| \( w_9 \) | \( \{(x_1y_4)(\partial x_0/\partial z) - y_4(\partial x_1/\partial z)\} \) |
| \( w_{10} \) | \( \{x_0x_2(\partial x_0/\partial z) - x_0x_1(\partial x_1/\partial z)\} \) |

Trivially, \( w_2 = 2w_{10}, w_4 = 2w_3, w_5 = 2w_7, w_6 = 2w_8, \) and in \( H^0(C, K^2L) \) we have

\[
x_0w_2 = 2x_1w_1, \quad x_1w_2 = 2x_0w_3, \quad x_0w_5 = 2x_1w_3, \quad x_0w_6 = x_1w_5 \quad \text{and} \quad x_1w_6 = 2x_0w_9.
\]

This clearly implies there can be no linear relation among \( \{w_1, w_2, w_3, w_5, w_6, w_8\} \) in \( H^0(C, K^2) \). Consequently, the kernel of \( \mu_1 \) has basis \( \{w_2 - 2w_{10}, w_4 - 2w_3, w_5 - 2w_7, w_6 - 2w_8\} \). By equation (*) we finally have

\[
\dim T_1^0(\mathcal{O}^2_{6,10}) = 3g - 3 + \rho + \dim(\ker \mu_1) = 3(10) - 3 + 10 - 3(10 - 6 + 2) + 4 = 23.
\]

It is easy to compute the dimension of the hyper-elliptic component \( W \) of \( \mathcal{O}^2_{6,10} \). Since the \( g^1_2 \) on a hyper-elliptic curve is unique and \( l = L \to C \), where \( |L| = 2g^1_2 + P + Q \), one has immediately that

\[
\dim W = \dim\{\text{genus 10, hyper-elliptic curves}\} + 2.
\]

By Hurwitz's formula,

\[
\dim\{\text{hyper-elliptic curves of genus } g\} = 2g - 1.
\]

Therefore,

\[
\dim W = 19 + 2 = 21.
\]
Thus, since \( \dim T_l(\mathcal{H}_{6,10}^2) = 23 > \dim T_l(\mathcal{H}_{6,10}^3) = 21 \), we conclude

**Theorem.** The "component of hyper-elliptics" in \( \mathcal{H}_{6,10}^2 \) is nonreduced.

A final remark: In general the singularities of \( \mathcal{H}_{d,g}^r \) are "worst" along \( \mathcal{H}_{d,g}^{r+1} \), and in fact one can show, using these methods, that if \( l \in \mathcal{H}_{6,10}^3 \),

\[
\dim T_l(\mathcal{H}_{6,10}^3) > 23.
\]

In spite of this, \( \mathcal{H}_{6,10}^3 \) is smooth, i.e.

\[
\dim T_l(\mathcal{H}_{6,10}^3) = \dim T_l(\mathcal{H}_{6,10}^3) = 19
\]

for all \( l \in \mathcal{H}_{6,10}^3 \). So it is very important to distinguish between \( T_l(\mathcal{H}_{d,g}^{r+1}) \) and \( T_l(\mathcal{H}_{d,g}^r) \) when \( l \in \mathcal{H}_{d,g}^{r+1} \subset \mathcal{H}_{d,g}^r \).

**References**


[Griffin 2] ______, *The component structure of \( \mathbb{H}_g^2 \) (to appear).*

**Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138**

**Current address:** Department of Mathematics, University of California, Los Angeles, California 90024