DILATIONS OF V-BOUNDED STOCHASTIC PROCESSES
INDEXED BY A LOCALLY COMPACT GROUP

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Abstract. It is proved that a stochastic process (i.e., a Hilbert space valued
function) indexed by a locally compact group is V-bounded (i.e., weakly har-
onizable in an appropriate sense) if, and only if, it can be expressed as
an orthogonal projection of a process whose covariance function R satisfies
R(s,t) = ρ(t⁻¹s) + ρ(st⁻¹) for some continuous positive-definite function ρ.
The result generalizes a well-known theorem due to H. Niemi, and depends on
the noncommutative Grothendieck type inequality of G. Pisier.

Let G be a locally compact (not necessarily Abelian) group and H a (complex)
Hilbert space. In analogy with the case G = R and H = L²(Ω, A, P) = {f ∈
L²[∫ f dP = 0]} for a probability space (Ω, A, P), we call any function ϕ: G —> H
a (stochastic) process. A process ϕ: G —> H is called V-bounded, if it is weakly
continuous and the set {∫ ϕdμ|μ ∈ L¹(G), ||μ||' ≤ 1} is bounded in H, where ||μ||'
ds the least upper bound of ||π(μ)|| when π ranges over the continuous unitary
representations of G. As shown in [6], this is equivalent to a property that might be
called weak harmonizability as in the commutative case, i.e., that there is a bounded
linear map Φ: C*(G) —> H whose Fourier transform ψ equals ϕ. Here ψ is defined by
the formula ψ(s) = Φ**(ω(s)) where Φ**: C*(G)** —> H is the bitranspose of Φ and
ω: G —> C*(G)** ⊂ L(Hω) is the representation of G corresponding to the universal
representation of the group C*-algebra C*(G) of G. For a detailed discussion of
these matters we refer to [6].

In [4], H. Niemi proved that in the case G = R a process is V-bounded if,
and only if, it is an orthogonal projection of a stationary process. For subsequent
developments of the theme one may, e.g., consult [1] and its bibliography. In this
note we extend Niemi’s result to the noncommutative setting. Our treatment is
also in part influenced by [1 and 2]. Let us begin with a general lemma, which
explains the role of 2-majorizability, inherent in the subject.

Lemma 1. Let A be an involutive algebra, Φ: A —> H a linear map and f₁, f₂: A —> C
positive linear forms (i.e., f_j(x*x) ≥ 0 for all x ∈ A, j = 1, 2). The following two
conditions are equivalent.

(i) ||Φx||² ≤ f₁(x*x) + f₂(xx*) for all x ∈ A.
(ii) There is a Hilbert space K such that for some linear map Ψ: A —> K and some
isometric linear map V: H —> K, Φ = V*Ψ and (Ψx|Ψy) = f₁(y*x) + f₂(xy*)
whenever x, y ∈ A.

Proof. First assume (i). Then (ii) can be proved along the lines of [2, pp. 256–
257]. Alternatively, denote h(x, y) = f₁(y*x) + f₂(xy*) and N = {x ∈ A|h(x, x) = 0}.

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Since $h$ is a positive Hermitian form, $N$ is a linear subspace of $A$, and $A/N$ can be completed to a Hilbert space $W$ with inner product $(x + N|y + N) = h(x, y)$. Now (i) implies the existence of a linear contraction $\Phi_0 : W \to H$ with $\Phi_0(x + N) = \Phi x$. Then as in [1, p. 274] an idea going back to [3] yields a Hilbert space $K$ along with isometric linear maps $\Psi_0 : W \to K$ and $V : H \to K$ such that $\Phi_0 = V^*\Psi_0$. Defining $\Psi x = \Psi_0(x + N)$ we get the $\Psi$ required for (ii); since $\Psi_0$ preserves inner products, $(\Psi x|\Psi y) = h(x, y)$. It is, conversely, clear that (ii) implies (i), since $||\Phi|| \leq ||\Psi||$.

As usual, $B(G)$ will stand for the vector space spanned by the continuous positive-definite functions on $G$. As in [6, p. 359], we denote by $T : B(G) \to C^*(G)$ the bijection satisfying $\rho(s) = (\omega(s), T\rho)$ for $s \in G$.

**Lemma 2.** A process $\phi : G \to H$ is V-bounded, if there are two functions $\rho_1, \rho_2 \in B(G)$ such that

\[(\phi(s)|\phi(t)) = \rho_1(t^{-1}s) + \rho_2(st^{-1}) \quad \text{for all } s, t \in G.\]

**Proof.** From (1) it follows that $\phi$ is norm continuous. Denote $f_i = T\rho_i \in C^*(G)^*$, $i = 1, 2$. If $s_j \in G$, $\lambda_j \in \mathbb{C}$ and $\mu = \sum_{j=1}^{n} \lambda_j \delta_{s_j}$ is the corresponding linear combination of point measures, we have

\[
\sum_{j=1}^{n} \lambda_j \phi(s_j) = \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_j \bar{\lambda}_k [f_1(\omega(s_j^{-1})\omega(s_j)) + f_2(\omega(s_j)\omega(s_j^{-1}))]
\]

\[= f_1(\omega(\mu)^*\omega(\mu)) + f_2(\omega(\mu)\omega(\mu)^*) \leq (||f_1|| + ||f_2||)||\mu||^2
\]

(see e.g. [6, pp. 358–359]). Thus $\phi$ is V-bounded (see Remark 6.3 in [6, p. 379]).

We are now ready to prove our main result, which generalizes Theorem 11 in [4, p. 124].

**Theorem.** For a stochastic process $\phi : G \to H$ the following two conditions are equivalent.

(i) $\phi$ is V-bounded.

(ii) There is a Hilbert space $K$ and a process $\psi : G \to K$ such that, for some isometric linear map $V : H \to K$ and for some continuous positive-definite function $\rho : G \to \mathbb{C}$, $\phi = V^* \circ \psi$ and $(\psi(s)|\psi(t)) = \rho(t^{-1}s) + \rho(st^{-1})$ whenever $s, t \in G$.

**Proof.** If $\psi$ is as in (ii), it is V-bounded by Lemma 2, and so $\phi = V^* \circ \psi$ is V-bounded, too. Now assume (i). Remark 6.3 in [6, p. 379] shows that $\phi$ is the Fourier transform of some bounded linear map $\Phi : C^*(G) \to H$. There is a positive linear functional $f : C^*(G) \to \mathbb{C}$ such that $||\Phi x||^2 \leq f(x^*x + xx^*)$ for all $x \in G$ (see [5, p. 400]), and so Lemma 1 yields a Hilbert space $K$ and a linear map $\Psi : C^*(G) \to K$ such that $(\Psi x|\Psi y) = f(y^*x) + f(xy^*)$ for all $x, y \in C^*(G)$ and that for some isometric linear map $V : H \to K$ we have $\Phi = V^* \Psi$. Since $||\Psi x||^2 \leq 2||f||||x||^2$, $\Psi$ is bounded, and defining $\psi$ as its Fourier transform (see [6, p. 360]) we obviously have $\phi = V^* \circ \psi$.

**References**


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