SPECTRAL INCLUSION FOR SUBNORMAL n-TUPLES

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Abstract. Let $S$ be a subnormal operator on a Hilbert space and let $N$ be its minimal extension. Then a celebrated theorem due to P. Halmos asserts that $\text{Sp}(N) \subseteq \text{Sp}(S)$, denoting by $\text{Sp}$ the spectrum. This note contains a multidimensional version, with respect to Taylor’s joint spectrum, of this spectral inclusion theorem.

Recently R. Curto [1] has extended Halmos’ spectral inclusion theorem for subnormal operators to the case of $n$-tuples of doubly commuting subnormal operators. In this note we improve Curto’s result by removing the double commutativity assumption.

Let $S = (S_1, \ldots, S_n)$ be a subnormal $n$-tuple of commuting operators on a Hilbert space $\mathcal{H}$ (i.e. there exists a commuting $n$-tuple of normal operators which extends $S$). Then there exists a unique, up to isometric isomorphism, minimal extension of $S$. Let $\text{Sp}(S, \mathcal{K})$ denote Taylor’s joint spectrum of $S$ on $\mathcal{K}$.

Theorem. Let $S$ be a commuting subnormal $n$-tuple on $\mathcal{K}$ and let $N$ be its minimal normal extension to a Hilbert space $\mathcal{H}$. Then

$$\text{Sp}(N, \mathcal{H}) \subseteq \text{Sp}(S, \mathcal{H}).$$

Proof. It is enough to prove that $0 \notin \text{Sp}(S, \mathcal{K})$ implies $0 \notin \text{Sp}(N, \mathcal{K})$, or equivalently, by a Gelfand transformation argument, that $0 \notin \text{Sp}(|N|, \mathcal{K})$, where $|N|^2 = \sum_{i=1}^n N_i N_i^*$.

Suppose $0 \notin \text{Sp}(S, \mathcal{K})$. Then the operator $S: \mathcal{K}^n \to \mathcal{K}$ is onto, and after a homothety, one can suppose that

$$(\forall) \ h \in \mathcal{K}, \ (\exists) \ h_1, \ldots, h_n \in \mathcal{K} \text{ such that } \sum_{i=1}^n S_i h_i = h,$$

and

$$\sum_{i=1}^n \|h_i\|^2 \leqslant \|h\|^2.$$

Take the spectral measure $E$ of $N$ and let $\mathcal{E}$ be the space $E(\{z||z|\leqslant 1/2n\})\mathcal{K}$, which reduces the operators $N_i$. If we prove that $\mathcal{E} \perp \mathcal{K}$, then, by the minimality of the extension $N$, $\mathcal{E}$ must be 0, hence $|N|$ will be invertible.

Received by the editors December 17, 1982.

1980 Mathematics Subject Classification. Primary 47B20; Secondary 47A10, 47D99.

Key words and phrases. Subnormal operator, commuting $n$-tuple, Taylor’s joint spectrum, minimal normal extension.

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0002-9939/84 $1.00 + $.25 per page

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Let \( l \in \mathcal{L} \) and \( h \in \mathcal{H} \). By using (1) a finite number of times, one obtains

\[
\left\langle l, h \right\rangle = \left| \left( l, \sum_{1 \leq i_1, \ldots, i_p \leq n} S_{i_1} \cdots S_{i_p} h_{i_1, \ldots, i_p} \right) \right|
\]

\[
= \left| \left( l, \sum N_{i_1} \cdots N_{i_p} h_{i_1, \ldots, i_p} \right) \right| \leq \sum \| N_{i_1}^* \cdots N_{i_p}^* l \| \cdot \| h_{i_1, \ldots, i_p} \|
\]

\[
\leq \sum \| N_{i_1}^p l \| \cdot \| h_{i_1, \ldots, i_p} \| \leq \| l \| / (2n)^p \sum \| h_{i_1, \ldots, i_p} \|
\]

\[
\leq \left( \| l \| / (2n)^p \right)^{1/n} \left( \sum \| h_{i_1, \ldots, i_p} \|^2 \right)^{1/2} \leq \| l \| \cdot \| h \| \left( 1/2 \sqrt{n} \right)^p .
\]

By passing to the limit when \( p \to \infty \), \( \left\langle l, h \right\rangle = 0 \), and the proof is complete.

REFERENCES


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