SPECTRAL INCLUSION FOR SUBNORMAL $n$-TUPLES

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Abstract. Let $S$ be a subnormal operator on a Hilbert space and let $N$ be its minimal extension. Then a celebrated theorem due to P. Halmos asserts that $\text{Sp}(N) \subseteq \text{Sp}(S)$, denoting by $\text{Sp}$ the spectrum. This note contains a multidimensional version, with respect to Taylor's joint spectrum, of this spectral inclusion theorem.

Recently R. Curto [1] has extended Halmos' spectral inclusion theorem for subnormal operators to the case of $n$-tuples of doubly commuting subnormal operators. In this note we improve Curto's result by removing the double commutativity assumption.

Let $S = (S_1, \ldots, S_n)$ be a subnormal $n$-tuple of commuting operators on a Hilbert space $\mathcal{H}$ (i.e. there exists a commuting $n$-tuple of normal operators which extends $S$). Then there exists a unique, up to isometric isomorphism, minimal extension of $S$. Let $\text{Sp}(S, \mathcal{K})$ denote Taylor's joint spectrum of $S$ on $\mathcal{K}$.

Theorem. Let $S$ be a commuting subnormal $n$-tuple on $\mathcal{K}$ and let $N$ be its minimal normal extension to a Hilbert space $\mathcal{K}$. Then

$$\text{Sp}(N, \mathcal{K}) \subseteq \text{Sp}(S, \mathcal{K}).$$

Proof. It is enough to prove that $0 \notin \text{Sp}(S, \mathcal{K})$ implies $0 \notin \text{Sp}(N, \mathcal{K})$, or equivalently, by a Gelfand transformation argument, that $0 \notin \text{Sp}(|N|, \mathcal{K})$, where $|N|^2 = \sum_{i=1}^n N_i N^*_i$.

Suppose $0 \notin \text{Sp}(S, \mathcal{K})$. Then the operator $S: \mathcal{K}^n \to \mathcal{K}$ is onto, and after a homothety, one can suppose that

$$(\forall) \ h \in \mathcal{K}, \ (\exists) \ h_1, \ldots, h_n \in \mathcal{K} \text{ such that } \sum_{i=1}^n S_i h_i = h,$$

and

$$\sum_{i=1}^n \| h_i \|^2 \leq \| h \|^2.$$

Take the spectral measure $E$ of $N$ and let $\mathcal{E}$ be the space $E(\{ z \mid |z| \leq 1/2n \}) \mathcal{K}$, which reduces the operators $N_i$. If we prove that $\mathcal{E} \perp \mathcal{K}$, then, by the minimality of the extension $N$, $\mathcal{E}$ must be 0, hence $|N|$ will be invertible.
Let $l \in \mathcal{L}$ and $h \in \mathcal{H}$. By using (1) a finite number of times, one obtains

$$\left| \langle l, h \rangle \right| = \left| \left( l, \sum_{1 \leq i_1, \ldots, i_p \leq n} S_{i_1} \cdots S_{i_p} h_{i_1, \ldots, i_p} \right) \right|$$

$$= \left| \left( l, \sum N_{i_1} \cdots N_{i_p} h_{i_1, \ldots, i_p} \right) \right| \leq \sum \| N_{i_1}^* \cdots N_{i_p}^* l \| \cdot \| h_{i_1, \ldots, i_p} \|$$

$$\leq \sum \| N_{i_1}^p l \| \cdot \| h_{i_1, \ldots, i_p} \| \leq \| l \| / (2n)^p \sum \| h_{i_1, \ldots, i_p} \|$$

$$\leq (\| l \| / (2n)^p) \sqrt{\frac{n^p}{2}} \left( \sum \| h_{i_1, \ldots, i_p} \|^2 \right)^{1/2} \leq \| l \| \cdot \| h \| \left( 1/2 \sqrt{n} \right)^p.$$  

By passing to the limit when $p \to \infty$, $\langle l, h \rangle = 0$, and the proof is complete.

**REFERENCES**