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SPECTRAL INCLUSION FOR SUBNORMAL n-TUPLES

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ABSTRACT. Let S be a subnormal operator on a Hilbert space and let N be its minimal extension. Then a celebrated theorem due to P. Halmos asserts that $\mathrm{Sp}(N) \subset \mathrm{Sp}(S)$, denoting by Sp the spectrum. This note contains a multidimensional version, with respect to Taylor's joint spectrum, of this spectral inclusion theorem.

Recently R. Curto [**1**] has extended Halmos' spectral inclusion theorem for subnormal operators to the case of n-tuples of doubly commuting subnormal operators. In this note we improve Curto's result by removing the double commutativity assumption.

Let $S = (S_1, \ldots, S_n)$ be a subnormal n-tuple of commuting operators on a Hilbert space $\mathcal{H}$ (i.e. there exists a commuting n-tuple of normal operators which extends S). Then there exists a unique, up to isometric isomorphism, minimal extension of S. Let $\mathrm{Sp}(S, \mathcal{H})$ denote Taylor's joint spectrum of S on $\mathcal{H}$.

THEOREM. *Let S be a commuting subnormal n-tuple on $\mathcal{H}$ and let N be its minimal normal extension to a Hilbert space $\mathcal{K}$. Then*

$$\mathrm{Sp}(N, \mathcal{K}) \subset \mathrm{Sp}(S, \mathcal{H}).$$

PROOF. It is enough to prove that $0 \notin \mathrm{Sp}(S, \mathcal{H})$ implies $0 \notin \mathrm{Sp}(N, \mathcal{K})$, or equivalently, by a Gelfand transformation argument, that $0 \notin \mathrm{Sp}(|N|, \mathcal{K})$, where $|N|^2 = \sum_{i=1}^n N_i N_i^*$.

Suppose $0 \notin \mathrm{Sp}(S, \mathcal{H})$. Then the operator $S\colon \mathcal{H}^n \to \mathcal{H}$ is onto, and after a homothety, one can suppose that

$$(1) \qquad \begin{aligned} &(\forall)\, h \in \mathcal{H},\ (\exists)\, h_1, \ldots, h_n \in \mathcal{H} \text{ such that } \sum_{i=1}^n S_i h_i = h, \\ &\text{and } \sum_{i=1}^n \|h_i\|^2 \leq \|h\|^2. \end{aligned}$$

Take the spectral measure E of N and let $\mathcal{L}$ be the space $E(\{z \mid |z| \leq 1/2n\})\mathcal{K}$, which reduces the operators N_i. If we prove that $\mathcal{L} \perp \mathcal{H}$, then, by the minimality of the extension N, $\mathcal{L}$ must be 0, hence $|N|$ will be invertible.

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Let $l \in \mathcal{L}$ and $h \in \mathcal{H}$. By using (1) a finite number of times, one obtains

$$\begin{aligned}
|\langle l, h\rangle| &= \left|\left\langle l, \sum_{1\leqslant i_1,\ldots,i_p\leqslant n} S_{i_1}\cdots S_{i_p}h_{i_1\cdots i_p}\right\rangle\right| \\
&= \left|\left\langle l, \sum N_{i_1}\cdots N_{i_p}h_{i_1\cdots i_p}\right\rangle\right| \leqslant \sum \left\|N_{i_1}^*\cdots N_{i_p}^* l\right\| \cdot \left\|h_{i_1\cdots i_p}\right\| \\
&\leqslant \sum \left\||N|^p l\right\| \cdot \left\|h_{i_1\cdots i_p}\right\| \leqslant \|l\|/(2n)^p \sum \left\|h_{i_1\cdots i_p}\right\| \\
&\leqslant \left(\|l\|/(2n)^p\right)\sqrt{n^p}\left(\sum \left\|h_{i_1\cdots i_p}\right\|^2\right)^{1/2} \leqslant \|l\|\cdot\|h\|\left(1/2\sqrt{n}\right)^p.
\end{aligned}$$

By passing to the limit when $p \to \infty$, $\langle l, h\rangle = 0$, and the proof is complete.

References

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