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A GENERAL ISEPIPHANIC INEQUALITY

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ABSTRACT. An inequality of Petty regarding the volume of a convex body and that of the polar of its projection body is shown to lead to an inequality between the volume of a convex body and the power means of its brightness function. A special case of this power-mean inequality is the classical isepiphanic (isoperimetric) inequality. The power-mean inequality can also be used to obtain strengthened forms and extensions of some known and conjectured geometric inequalities. Affine projection measures (Quermassintegrale) are introduced.

In [**12**] it was shown that the Blaschke-Santaló inequality [**23**] leads immediately to a power-mean inequality relating the volume of a convex body and the power means of its width (function). Special cases of this power-mean inequality include the classical inequalities of Urysohn and Bieberbach.

It will be shown in the present note that an inequality of Petty [**19**], which we will refer to as the Petty projection inequality, leads immediately to an analogous power-mean inequality relating the volume of a convex body and the power means of its brightness (function). A special case of this power-mean inequality is the classical isepiphanic (isoperimetric) inequality. This power-mean inequality also leads to inequalities similar to some width-volume inequalities obtained by Chakerian [**6**, **7**], Chakerian and Sangwine-Yager [**8**], and the author [**15**]. When combined with other known inequalities, this power-mean inequality can be used to obtain a strengthened form of an inequality of Knothe [**11**] and Chakerian [**5**] relating the volume of a convex body and the arithmetic mean of the volumes of its circumscribed right cylinders. Finally, it solves completely a problem posed in [**14**], and can be used to prove two (similar) conjectures of the author [**16**, **26**].

The setting for this note is Euclidean n-dimensional space, $\mathbf{R}^n$ $(n \geq 2)$. We will use the letter K (possibly with subscripts) to denote a convex body (compact convex set with nonempty interior) in $\mathbf{R}^n$. We use S^{n-1} to denote the surface and ω_n to denote the n-dimensional volume of the unit ball in $\mathbf{R}^n$. The letter u will denote a unit vector, exclusively. For a given direction $u \in S^{n-1}$, we use E_u to denote the hyperplane (passing through the origin) orthogonal to u. For a given K and $u \in S^{n-1}$, we use $b_K(u)$ and $\sigma_K(u)$ to denote respectively the width and brightness of K in the direction u; i.e., $\sigma_K(u)$ is the $(n-1)$-dimensional volume of the projection of K onto E_u, while $b_K(u)$ is the 1-dimensional volume of the projection of K onto the orthogonal complement of E_u. For the volume, surface area, and mean width of K, we write $V(K)$, $S(K)$, and $B(K)$, respectively. The reader is referred to [**3** and **9**] for material relating to convex bodies.

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For a positive continuous function f defined on S^{n-1} and a real number $p \neq 0$, the *p-mean* of f, $M_p[f]$, is defined by

$$M_p[f] = \left[\frac{1}{n\omega_n}\int_{S^{n-1}} f^p(u)\,dS(u)\right]^{1/p},$$

where $dS(u)$ is the surface area element on S^{n-1} at u. For $p = -\infty$, 0, or ∞, $M_p[f]$ is defined by

$$M_p[f] = \lim_{s\to p} M_s[f].$$

It is well known [**10**, p. 143] that $M_p[f]$ is continuous in p, and that

$$M_\infty[f] = \max\{f(u) \mid u \in S^{n-1}\},$$

while,

$$M_{-\infty}[f] = \min\{f(u) \mid u \in S^{n-1}\}.$$

If $p < q$, then we have Jensen's inequality [**10**, p. 144],

$$M_p[f] \le M_q[f], \tag{1}$$
with equality if and only if f is constant.

For a convex body K in $\mathbf{R}^n$ and a point $*$ in the interior of K, let K^* denote the polar reciprocal body of K with respect to the unit sphere centered at $*$. The Blaschke-Santaló inequality is

$$\operatorname{Inf} V(K)V(K^*) \le \omega_n^2, \tag{2}$$
with equality if and only if K is an ellipsoid,

where the Inf is taken over all points $*$ in the interior of K. The inequality is due to Blaschke [**1**] for $n \le 3$ and to Santaló [**23**] for $n \ge 2$ (see also the comments in Schneider [**25**, p. 552]). From these works also follow the conditions for equality when K is assumed to be sufficiently smooth. The conditions for equality for arbitrary convex bodies were recently obtained by Petty [**21**]. (See also Saint Raymond [**22**] for the case where K is assumed centrally symmetric.)

In [**12**] it was shown that a direct consequence of (2) is

THEOREM 1. *For $p > -n$ and for all convex bodies K in $\mathbf{R}^n$*

$$[V(K)/\omega_n]^{1/n} \le M_p[b_K/2],$$
with equality if and only if K is a ball.

If $p = -n$, the inequality remains valid; however, equality can occur if and only if K is an ellipsoid. For $p < -n$, the inequality does not hold (for all K).

As noted in [**12**], since $M_\infty[b_K/2]$ is one-half the diameter of K, the case $p = \infty$ in Theorem 1 is the Bieberbach inequality [**9**, p. 173], and since $M_1[b_K/2]$ is equal to $B(K)/2$, the case $p = 1$ is the Urysohn inequality [**3**, p. 76].

Clearly, from (1), it follows that for $p < q$ we have

$$M_p[b_K/2] \le M_q[b_K/2], \tag{3}$$
with equality if and only if K is of constant width.

From (3), we see that, in Theorem 1, larger values of p result in 'weaker' inequalities.

As will be shown presently, the Petty projection inequality leads immediately to a result analogous to Theorem 1:

THEOREM 2. *For $p > -n$ and for all convex bodies K in $\mathbf{R}^n$*

$$[V(K)/\omega_n]^{(n-1)/n} \leq M_p[\sigma_K/\omega_{n-1}],$$

with equality if and only if K is a ball.

For $p = -n$, the inequality remains valid; however, equality can occur if and only if K is an ellipsoid. For $p < -n$, the inequality does not hold (for all K).

Since the Cauchy surface area formula [**9**, p. 208] is

$$S(K) = n\omega_n M_1[\sigma_K/\omega_{n-1}],$$

the case $p = 1$ in Theorem 2 is the classical isepiphanic (isoperimetric) inequality

$$n\omega_n^{1/n} V(K)^{(n-1)/n} \leq S(K),$$

with equality if and only if K is a ball.

From (1), it follows immediately that for $p < q$ we have

$$M_p[\sigma_K/\omega_{n-1}] \leq M_q[\sigma_K/\omega_{n-1}], \tag{4}$$

with equality if and only if K is of constant brightness.

From (4), we see that, in Theorem 2, larger values of p result in 'weaker' inequalities. We note that inequality (4) for the case $p = -n$, $q = 1$ can be found in Petty [**19**, p. 40].

We now prove Theorem 2. For a given convex body K, the projection body of K, ΠK is defined [**3**, p. 45] (see also Bolker [**2**] and Schneider–Weil [**26**]) as the convex body whose supporting hyperplane in a given direction u has a distance $\sigma_K(u)$ from the origin; i.e., the support function of ΠK is σ_K. The Petty projection inequality [**19**, p. 40] is

$$I_m(\Pi K)V(K)^{n-1} \leq (\omega_n/\omega_{n-1})^n,$$

with equality if and only if K is an ellipsoid,

where $I_m(\Pi K)$ denotes the minimum of the volumes of the polar reciprocal bodies of ΠK. Since ΠK is centrally symmetric, it follows (see, for example, [**13**, **19**, **20**, **23**]) that

$$I_m(\Pi K) = V(\Pi^\circ K),$$

where $\Pi^\circ K$ denotes the polar reciprocal of ΠK with respect to (the unit sphere centered at) the origin. Since the boundary of $\Pi^\circ K$ can be represented in polar form by $r = \sigma_K(u)^{-1}$, the volume of $\Pi^\circ K$ is given by

$$V(\Pi^\circ K) = \frac{1}{n}\int_{S^{n-1}} \sigma_K^{-n}(u)\, dS(u).$$

It follows that the Petty projection inequality is the case $p = -n$ in Theorem 2. The cases where $p > -n$, now, follow from the case $p = -n$ if we use (4). To see that the inequality (in Theorem 2) does not hold for $p < -n$, take K to be any nonspherical ellipsoid, and use the case $p = -n$ in conjunction with (4).

We note that Theorem 2 completely solves the problem posed in [**14**]. We also note that for $n = 2$ (the plane case) both theorems coincide.

As will be seen shortly, the case $p = -1$ in Theorem 2 is of particular interest.

The projection measures (Quermassintegrale) $W_0, W_1, \ldots, W_n$ in $\mathbf{R}^n$ can be defined (see [**9**, p. 234]) by letting $W_0(K) = V(K)$, $W_n(K) = \omega_n$, and, for $0 < i < n$, letting

$$\frac{\omega_i}{\omega_n} W_{n-i}(K) = \frac{\omega_{n-i}}{\omega_n c_{in}} \int V_i(K \mid E_i) d\overline{\overline{E}}_i,$$

where all such integrals are to be taken over the entire space of freely rotating i-dimensional flats E_i through the origin, $K \mid E_i$ denotes the projection of K onto E_i, V_i denotes i-dimensional volume and $d\overline{\overline{E}}_i$ is the rotation density, normalized so that

$$\int d\overline{\overline{E}}_i = \frac{\omega_n c_{in}}{\omega_{n-i}},$$

where

$$c_{in} = \binom{n}{i} \frac{\omega_{n-1} \cdots \omega_{n-i}}{\omega_1 \cdots \omega_i}.$$

The harmonic projection measures (harmonische Quermassintegrale) $\tilde{W}_0, \tilde{W}_1, \ldots, \tilde{W}_n$ in $\mathbf{R}^n$ are defined by Hadwiger [**9**, p. 267] by letting $\tilde{W}_0(K) = V(K)$, $\tilde{W}_n(K) = \omega_n$, and, for $0 < i < n$, letting

$$\frac{\omega_i}{\omega_n} \tilde{W}_{n-i}(K) = \left[\frac{\omega_{n-i}}{\omega_n c_{in}} \int V_i(K \mid E_i)^{-1} d\overline{\overline{E}}_i \right]^{-1}.$$

It follows (see [**9**, p. 267]) that

$$\tilde{W}_i(K) \leq W_i(K),$$

with equality for $0 < i < n$ if and only if the $(n-i)$-dimensional projections of K have constant $(n-i)$-dimensional volume.

Obviously, we have

$$W_{n-1}(K) = \omega_n M_1[b_K/2] \quad \text{and} \quad \tilde{W}_{n-1}(K) = \omega_n M_{-1}[b_K/2],$$

while

$$W_1(K) = \omega_n M_1[\sigma_K/\omega_{n-1}] \quad \text{and} \quad \tilde{W}_1(K) = \omega_n M_{-1}[\sigma_K/\omega_{n-1}].$$

The case $p = 1$ in Theorem 1 is the Urysohn inequality,

$$\omega_n^{n-1} V(K) \leq W_{n-1}(K)^n,$$

with equality if and only if K is a ball,

while the case $p = -1$ in Theorem 1 is the stronger harmonic Urysohn inequality (see [**12**, **16**]),

$$\omega_n^{n-1} V(K) \leq \tilde{W}_{n-1}(K)^n,$$

with equality if and only if K is a ball.

Similarly, the case $p = 1$ in Theorem 2 is the isepiphanic inequality,

$$\omega_n V(K)^{n-1} \leq W_1(K)^n,$$

with equality if and only if K is a ball,

while the case $p = -1$ in Theorem 2 is the stronger harmonic isepiphanic inequality,

$$\omega_n V(K)^{n-1} \leq \tilde{W}_1(K)^n,$$

with equality if and only if K is a ball.

This last inequality is conjectured in [**16**, p. 147].

In light of the critical role played by the case $p = -n$ in both theorems, one is led to define affine projection measures $\Phi_0, \Phi_1, \ldots, \Phi_n$ in $\mathbf{R}^n$ by taking $\Phi_0(A) = V(A)$, $\Phi_n(A) = \omega_n$, and, for $0 < i < n$, letting

$$\frac{\omega_i}{\omega_n}\Phi_{n-i}(K) = \left[\frac{\omega_{n-i}}{\omega_n c_{in}}\int V_i(K \mid E_i)^{-n}\,d\overline{E}_i\right]^{-1/n}.$$

Obviously, we have

$$\Phi_i(K) \le \tilde{W}_i(K) \le W_i(K),$$

with equality for $0 < i < n$ if and only if the $(n-i)$-dimensional projections of K have constant $(n-i)$-dimensional volume.

As noted by Hadwiger [**9**, p. 267], the harmonic projection measure $\tilde{W}_i$ (viewed as a functional on the space of convex bodies in $\mathbf{R}^n$, endowed with the topology induced by the Hausdorff metric [**9**, p. 151]) is positive, continuous, bounded, monotone (increasing), homogeneous of degree $n-i$, and invariant under motions (translations and rotations). It is easy to verify that the affine projection measure Φ_i has exactly the same properties.

Hadwiger [**9**, p. 268] proves that for the Minkowski (vector) sum $K_1 + K_2$ one has

$$\tilde{W}_i(K_1 + K_2)^{1/(n-i)} \ge \tilde{W}_i(K_1)^{1/(n-i)} + \tilde{W}_i(K_2)^{1/(n-i)};$$

i.e., $\tilde{W}_i^{1/(n-i)}$ is concave. Similarly, following in the same manner as Hadwiger, one has

$$\Phi_i(K_1 + K_2)^{1/(n-i)} \ge \Phi_i(K_1)^{1/(n-i)} + \Phi_i(K_2)^{1/(n-i)}.$$

In terms of affine projection measures, the case $p = -n$ in Theorem 1 may be viewed as the affine Bieberbach inequality,

$$\omega_n^{n-1}V(K) \le \Phi_{n-1}(K)^n,$$

with equality if and only if K is an ellipsoid,

while the case $p = -n$ in Theorem 2 (the Petty projection inequality) may be viewed as the affine isepiphanic inequality,

$$\omega_n V(K)^{n-1} \le \Phi_1(K)^n,$$

with equality if and only if K is an ellipsoid.

For a given convex body K in $\mathbf{R}^n$ and a direction $u \in S^{n-1}$ let $l_K(u,x)$ denote the length of the cord of K that is orthogonal to E_u and (when extended) passes through the point $x \in E_u$. Let $l_K(u)$ denote the mean length of chords of K that are in the direction u; i.e.,

$$l_K(u) = \frac{1}{\sigma_K(u)}\int_{K|E_u} l_K(u,x)\,dV_{n-1}(x),$$

where $dV_{n-1}(x)$ is the $(n-1)$-dimensional volume element on E_u at x. Clearly,

$$l_K(u) = V(K)\sigma_K(u)^{-1}. \tag{5}$$

The following 'dual' of the Urysohn inequality was conjectured by the author at the 1978 Oberwolfach 'Konvexe Körper' conference (see [**27**, p. 265]):

$$\frac{\omega_{n-1}}{n\omega_n}\int_{S^{n-1}} l_K(u)\,dS(u) \le \omega_n^{(n-1)/n}V(K)^{1/n},$$

with equality if and only if K is a ball.

This is an immediate consequence of the harmonic isepiphanic inequality (case $p = -1$ in Theorem 2) if we use (5).

The inequality in Theorem 2 with $p = -n$ (the Petty projection inequality) can be used to obtain brightness-volume inequalities analogous to some width-volume inequalities obtained by Chakerian [**6**, **7**], Chakerian and Sangwine-Yager [**8**], and Lutwak [**15**].

For convex bodies $K_1, \ldots, K_n$ in $\mathbf{R}^n$, and for a real number $p \neq 0$, we can define $S_p(K_1, \ldots, K_n)$ by:

$$S_p(K_1, \ldots, K_n) = \left[\frac{1}{n\omega_n} \int_{S^{n-1}} [\sigma_{K_1}(u) \cdots \sigma_{K_n}(u)]^p \, dS(u)\right]^{1/p}.$$

Following as in [**15**], we can use the Petty projection inequality and the Hölder inequality [**10**, p. 140] to obtain

$$(\omega_{n-1}^n/\omega_n^{n-1})[V(K_1)\cdots V(K_n)]^{(n-1)/n} \leq S_{-1}(K_1, \ldots, K_n), \tag{6}$$
with equality if and only if the K_i are homothetic ellipsoids.

The inequality (6) is a strengthened form of

$$(\omega_{n-1}^n/\omega_n^{n-1})[V(K_1)\cdots V(K_n)]^{(n-1)/n} \leq S_1(K_1, \ldots, K_n),$$
with equality if and only if all K_i are balls.

In connection with the last two inequalities, we note that a centrally symmetric body K always has volume greater than that of any other convex body whose brightness function is the same as that of K (see [**18** and **24**]).

If we combine (5) and (6) we obtain an inequality in the spirit of the concurrent cross-section inequality of Busemann [**4**] (also see [**17**]):

$$\frac{1}{n}\int_{S^{n-1}} l_{K_1}(u)\cdots l_{K_n}(u)\, dS(u) \leq (\omega_n^n/\omega_{n-1}^n)[V(K_1)\cdots V(K_n)]^{1/n},$$
with equality if and only if the K_i are homothetic ellipsoids.

For a convex body K and a direction $u \in S^{n-1}$, let $V_K(u)$ denote the volume of the right cylinder circumscribed about K whose generators are orthogonal to u. Clearly $V_K(u) = b_K(u)\sigma_K(u)$. By using the inequalities of Theorems 1 and 2 in conjunction with the Hölder inequality we obtain

$$\frac{2\omega_{n-1}}{\omega_n} V(K) \leq \left[\frac{1}{n\omega_n}\int_{S^{n-1}} V_K(u)^{-1}\, dS(u)\right]^{-1},$$
with equality if and only if K is a ball.

This is a strengthened form of the inequality

$$\frac{2\omega_{n-1}}{\omega_n} V(K) \leq \frac{1}{n\omega_n}\int_{S^{n-1}} V_K(u)\, dS(u),$$
with equality if and only if K is a ball,

which was obtained by Knothe [**11**] for $n = 3$ and proved by Chakerian [**5**] for $n \geq 3$.

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