A GENERAL ISEPIPHANIC INEQUALITY

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Abstract. An inequality of Petty regarding the volume of a convex body and that of the polar of its projection body is shown to lead to an inequality between the volume of a convex body and the power means of its brightness function. A special case of this power-mean inequality is the classical isepiphanic (isoperimetric) inequality. The power-mean inequality can also be used to obtain strengthened forms and extensions of some known and conjectured geometric inequalities. Affine projection measures (Quermassintegrale) are introduced.

In [12] it was shown that the Blaschke-Santaló inequality [23] leads immediately to a power-mean inequality relating the volume of a convex body and the power means of its width (function). Special cases of this power-mean inequality include the classical inequalities of Urysohn and Bieberbach.

It will be shown in the present note that an inequality of Petty [19], which we will refer to as the Petty projection inequality, leads immediately to an analogous power-mean inequality relating the volume of a convex body and the power means of its brightness (function). A special case of this power-mean inequality is the classical isepiphanic (isoperimetric) inequality. This power-mean inequality also leads to inequalities similar to some width-volume inequalities obtained by Chakerian [6, 7], Chakerian and Sangwine-Yager [8], and the author [15]. When combined with other known inequalities, this power-mean inequality can be used to obtain a strengthened form of an inequality of Knothe [11] and Chakerian [5] relating the volume of a convex body and the arithmetic mean of the volumes of its circumscribed right cylinders. Finally, it solves completely a problem posed in [14], and can be used to prove two (similar) conjectures of the author [16, 26].

The setting for this note is Euclidean $n$-dimensional space, $\mathbb{R}^n$ ($n \geq 2$). We will use the letter $K$ (possibly with subscripts) to denote a convex body (compact convex set with nonempty interior) in $\mathbb{R}^n$. We use $S^{n-1}$ to denote the surface and $\omega_n$ to denote the $n$-dimensional volume of the unit ball in $\mathbb{R}^n$. The letter $u$ will denote a unit vector, exclusively. For a given direction $u \in S^{n-1}$, we use $E_u$ to denote the hyperplane (passing through the origin) orthogonal to $u$. For a given $K$ and $u \in S^{n-1}$, we use $b_K(u)$ and $\sigma_K(u)$ to denote respectively the width and brightness of $K$ in the direction $u$; i.e., $\sigma_K(u)$ is the $(n-1)$-dimensional volume of the projection of $K$ onto $E_u$, while $b_K(u)$ is the 1-dimensional volume of the projection of $K$ onto the orthogonal complement of $E_u$. For the volume, surface area, and mean width of $K$, we write $V(K)$, $S(K)$, and $B(K)$, respectively. The reader is referred to [3 and 9] for material relating to convex bodies.
For a positive continuous function \( f \) defined on \( S^{n-1} \) and a real number \( p \neq 0 \), the \( p \)-mean of \( f \), \( M_p[f] \), is defined by

\[
M_p[f] = \left[ \frac{1}{n \omega_n} \int_{S^{n-1}} f^p(u) \, dS(u) \right]^{1/p},
\]
where \( dS(u) \) is the surface area element on \( S^{n-1} \) at \( u \). For \( p = -\infty, 0, \) or \( \infty \), \( M_p[f] \) is defined by

\[
M_p[f] = \lim_{s \to p} M_s[f].
\]

It is well known [10, p. 143] that \( M_p[f] \) is continuous in \( p \), and that

\[
M_\infty[f] = \max\{ f(u) \mid u \in S^{n-1} \},
\]
while,

\[
M_{-\infty}[f] = \min\{ f(u) \mid u \in S^{n-1} \}.
\]

If \( p < q \), then we have Jensen’s inequality [10, p. 144],

\[
M_p[f] \leq M_q[f],
\]
with equality if and only if \( f \) is constant.

For a convex body \( K \) in \( \mathbb{R}^n \) and a point \( * \) in the interior of \( K \), let \( K^* \) denote the polar reciprocal body of \( K \) with respect to the unit sphere centered at \( * \). The Blaschke-Santaló inequality is

\[
\inf V(K)V(K^*) \leq \omega_n^2,
\]
with equality if and only if \( K \) is an ellipsoid,

where the Inf is taken over all points \( * \) in the interior of \( K \). The inequality is due to Blaschke [1] for \( n \leq 3 \) and to Santaló [23] for \( n \geq 2 \) (see also the comments in Schneider [25, p. 552]). From these works also follow the conditions for equality when \( K \) is assumed to be sufficiently smooth. The conditions for equality for arbitrary convex bodies were recently obtained by Petty [21]. (See also Saint Raymond [22] for the case where \( K \) is assumed centrally symmetric.)

In [12] it was shown that a direct consequence of (2) is

**Theorem 1.** For \( p > -n \) and for all convex bodies \( K \) in \( \mathbb{R}^n \)

\[
\left[ V(K)/\omega_n \right]^{1/n} \leq M_p[b_K/2] ,
\]
with equality if and only if \( K \) is a ball.

If \( p = -n \), the inequality remains valid; however, equality can occur if and only if \( K \) is an ellipsoid. For \( p < -n \), the inequality does not hold (for all \( K \)).

As noted in [12], since \( M_\infty[b_K/2] \) is one-half the diameter of \( K \), the case \( p = \infty \) in Theorem 1 is the Bieberbach inequality [9, p. 173], and since \( M_1[b_K/2] \) is equal to \( B(K)/2 \), the case \( p = 1 \) is the Urysohn inequality [3, p. 76].

Clearly, from (1), it follows that for \( p < q \) we have

\[
M_p[b_K/2] \leq M_q[b_K/2],
\]
with equality if and only if \( K \) is of constant width.

From (3), we see that, in Theorem 1, larger values of \( p \) result in ‘weaker’ inequalities.

As will be shown presently, the Petty projection inequality leads immediately to a result analogous to Theorem 1:
Theorem 2. For \( p > -n \) and for all convex bodies \( K \) in \( \mathbb{R}^n \)

\[
[V(K)/\omega_n]^{n/(n-1)} \leq M_p[\sigma_K/\omega_{n-1}],
\]

with equality if and only if \( K \) is a ball.

For \( p = -n \), the inequality remains valid; however, equality can occur if and only if \( K \) is an ellipsoid. For \( p < -n \), the inequality does not hold (for all \( K \)).

Since the Cauchy surface area formula [9, p. 208]

\[
S(K) = n\omega_nM_1[\sigma_K/\omega_{n-1}],
\]

the case \( p = 1 \) in Theorem 2 is the classical isepiphanic (isoperimetric) inequality

\[
n\omega_n^{1/n}V(K)^{(n-1)/n} \leq S(K),
\]

with equality if and only if \( K \) is a ball.

From (1), it follows immediately that for \( p < q \) we have

\[
M_p[\sigma_K/\omega_{n-1}] \leq M_q[\sigma_K/\omega_{n-1}],
\]

with equality if and only if \( K \) is of constant brightness.

From (4), we see that, in Theorem 2, larger values of \( p \) result in ‘weaker’ inequalities.

We now prove Theorem 2. For a given convex body \( K \), the projection body of \( K \), \( \Pi K \) is defined [3, p. 45] (see also Bolker [2] and Schneider–Weil [26]) as the convex body whose supporting hyperplane in a given direction \( u \) has a distance \( \sigma_K(u) \) from the origin; i.e., the support function of \( \Pi K \) is \( \sigma_K \). The Petty projection inequality [19, p. 40] is

\[
I_m(\Pi K)V(K)^{n-1} \leq (\omega_n/\omega_{n-1})^n,
\]

with equality if and only if \( K \) is an ellipsoid,

where \( I_m(\Pi K) \) denotes the minimum of the volumes of the polar reciprocal bodies of \( \Pi K \). Since \( \Pi K \) is centrally symmetric, it follows (see, for example, [13, 19, 20, 23]) that

\[
I_m(\Pi K) = V(\Pi^o K),
\]

where \( \Pi^o K \) denotes the polar reciprocal of \( \Pi K \) with respect to (the unit sphere centered at) the origin. Since the boundary of \( \Pi^o K \) can be represented in polar form by \( r = \sigma_K(u)^{-1} \), the volume of \( \Pi^o K \) is given by

\[
V(\Pi^o K) = \frac{1}{n} \int_{S^{n-1}} \sigma^{-n}_K(u) dS(u).
\]

It follows that the Petty projection inequality is the case \( p = -n \) in Theorem 2. The cases where \( p > -n \), now, follow from the case \( p = -n \) if we use (4). To see that the inequality (in Theorem 2) does not hold for \( p < -n \), take \( K \) to be any nonspherical ellipsoid, and use the case \( p = -n \) in conjunction with (4).

We note that Theorem 2 completely solves the problem posed in [14]. We also note that for \( n = 2 \) (the plane case) both theorems coincide.

As will be seen shortly, the case \( p = -1 \) in Theorem 2 is of particular interest.
The projection measures (Quermassintegrale) \( W_0, W_1, \ldots, W_n \) in \( \mathbb{R}^n \) can be defined (see [9, p. 234]) by letting \( W_0(K) = V(K) \), \( W_n(K) = \omega_n \), and, for \( 0 < i < n \), letting

\[
\frac{\omega_i}{\omega_n} W_{n-i}(K) = \frac{\omega_{n-i}}{\omega_n c_{in}} \int V_i(K \mid E_i) \, d\overline{E}_i,
\]

where all such integrals are to be taken over the entire space of freely rotating \( i \)-dimensional flats \( E_i \) through the origin, \( K \mid E_i \) denotes the projection of \( K \) onto \( E_i \), \( V_i \) denotes \( i \)-dimensional volume and \( d\overline{E}_i \) is the rotation density, normalized so that

\[
\int d\overline{E}_i = \frac{\omega_n c_{in}}{\omega_{n-i}},
\]

where

\[
c_{in} = \binom{n}{i} \frac{\omega_{n-1} \cdots \omega_{n-i}}{\omega_1 \cdots \omega_i}.
\]

The harmonic projection measures (harmonische Quermassintegrale) \( \tilde{W}_0, \tilde{W}_1, \ldots, \tilde{W}_n \) in \( \mathbb{R}^n \) are defined by Hadwiger [9, p. 267] by letting \( \tilde{W}_0(K) = V(K) \), \( \tilde{W}_n(K) = \omega_n \), and, for \( 0 < i < n \), letting

\[
\frac{\omega_i}{\omega_n} \tilde{W}_{n-i}(K) = \left[ \frac{\omega_{n-i}}{\omega_n c_{in}} \int V_i(K \mid E_i)^{-1} \, d\overline{E}_i \right]^{-1}.
\]

It follows (see [9, p. 267]) that

\[
\tilde{W}_i(K) \leq W_i(K),
\]

with equality for \( 0 < i < n \) if and only if the \( (n - i) \)-dimensional projections of \( K \) have constant \( (n - i) \)-dimensional volume.

Obviously, we have

\[
W_{n-1}(K) = \omega_n M_1 [b_K / 2] \quad \text{and} \quad \tilde{W}_{n-1}(K) = \omega_n M_{-1} [b_K / 2],
\]

while

\[
W_1(K) = \omega_n M_1 [\sigma_K / \omega_{n-1}] \quad \text{and} \quad \tilde{W}_1(K) = \omega_n M_{-1} [\sigma_K / \omega_{n-1}].
\]

The case \( p = 1 \) in Theorem 1 is the Urysohn inequality,

\[
\omega_n V(K) \leq W_{n-1}(K)^n,
\]

with equality if and only if \( K \) is a ball,

while the case \( p = -1 \) in Theorem 1 is the stronger harmonic Urysohn inequality (see [12, 16]),

\[
\omega_n V(K)^{-1} \leq \tilde{W}_{n-1}(K)^n,
\]

with equality if and only if \( K \) is a ball.

Similarly, the case \( p = 1 \) in Theorem 2 is the isepiphanic inequality,

\[
\omega_n V(K)^{n-1} \leq W_1(K)^n,
\]

with equality if and only if \( K \) is a ball,

while the case \( p = -1 \) in Theorem 2 is the stronger harmonic isepiphanic inequality,

\[
\omega_n V(K)^{n-1} \leq \tilde{W}_1(K)^n,
\]

with equality if and only if \( K \) is a ball.
This last inequality is conjectured in [16, p. 147].

In light of the critical role played by the case \( p = -n \) in both theorems, one is led to define affine projection measures \( \Phi_0, \Phi_1, \ldots, \Phi_n \) in \( \mathbb{R}^n \) by taking \( \Phi_0(A) = V(A) \), \( \Phi_n(A) = \omega_n \), and, for \( 0 < i < n \), letting

\[
\frac{\omega_i}{\omega_n} \Phi_{n-i}(K) = \left[ \frac{\omega_{n-i}}{\omega_n} \int V_i(K \mid E_i)^{-n} dE_i \right]^{-1/n}.
\]

Obviously, we have

\[
\Phi_i(K) \leq \tilde{W}_i(K) \leq W_i(K),
\]

with equality for \( 0 < i < n \) if and only if the \((n-i)\)-dimensional projections of \( K \) have constant \((n-i)\)-dimensional volume.

As noted by Hadwiger [9, p. 267], the harmonic projection measure \( \tilde{W}_i \) (viewed as a functional on the space of convex bodies in \( \mathbb{R}^n \), endowed with the topology induced by the Hausdorff metric [9, p. 151]) is positive, continuous, bounded, monotone (increasing), homogeneous of degree \( n-i \), and invariant under motions (translations and rotations). It is easy to verify that the affine projection measure \( \Phi_i \) has exactly the same properties.

Hadwiger [9, p. 268] proves that for the Minkowski (vector) sum \( K_1 + K_2 \) one has

\[
\tilde{W}_i(K_1 + K_2)^{1/(n-i)} \geq \tilde{W}_i(K_1)^{1/(n-i)} + \tilde{W}_i(K_2)^{1/(n-i)},
\]

i.e., \( \tilde{W}_i^{1/(n-i)} \) is concave. Similarly, following in the same manner as Hadwiger, one has

\[
\Phi_i(K_1 + K_2)^{1/(n-i)} \geq \Phi_i(K_1)^{1/(n-i)} + \Phi_i(K_2)^{1/(n-i)}.
\]

In terms of affine projection measures, the case \( p = -n \) in Theorem 1 may be viewed as the affine Bieberbach inequality,

\[
\omega_n^{n-1} V(K) \leq \Phi_{n-1}(K)^n,
\]

with equality if and only if \( K \) is an ellipsoid,

while the case \( p = -n \) in Theorem 2 (the Petty projection inequality) may be viewed as the affine isepiphanic inequality,

\[
\omega_n V(K)^{n-1} \leq \Phi_1(K)^n,
\]

with equality if and only if \( K \) is an ellipsoid.

For a given convex body \( K \) in \( \mathbb{R}^n \) and a direction \( u \in S^{n-1} \) let \( l_K(u, x) \) denote the length of the cord of \( K \) that is orthogonal to \( E_u \) and (when extended) passes through the point \( x \in E_u \). Let \( l_K(u) \) denote the mean length of chords of \( K \) that are in the direction \( u \); i.e.,

\[
l_K(u) = \frac{1}{\sigma_K(u)} \int_{E_u} l_K(u, x) dV_{n-1}(x),
\]

where \( dV_{n-1}(x) \) is the \((n-1)\)-dimensional volume element on \( E_u \) at \( x \). Clearly,

\[
l_K(u) = V(K)\sigma_K(u)^{-1}.
\]

The following 'dual' of the Urysohn inequality was conjectured by the author at the 1978 Oberwolfach 'Konvexe Körper' conference (see [27, p. 265]):

\[
\frac{\omega_{n-1}}{n\omega_n} \int_{S^{n-1}} l_K(u) dS(u) \leq \omega_n^{(n-1)/n} V(K)^{1/n},
\]

with equality if and only if \( K \) is a ball.
This is an immediate consequence of the harmonic isepiphanic inequality (case $p = -1$ in Theorem 2) if we use (5).

The inequality in Theorem 2 with $p = -n$ (the Petty projection inequality) can be used to obtain brightness-volume inequalities analogous to some width-volume inequalities obtained by Chakerian [6, 7], Chakerian and Sangwine-Yager [8], and Lutwak [15].

For convex bodies $K_1, \ldots, K_n$ in $\mathbb{R}^n$, and for a real number $p \neq 0$, we can define $S_p(K_1, \ldots, K_n)$ by:

$$S_p(K_1, \ldots, K_n) = \left[ \frac{1}{n \omega_n} \int_{S^{n-1}} [\sigma_{K_1}(u) \cdots \sigma_{K_n}(u)]^p dS(u) \right]^{1/p}.$$

Following as in [15], we can use the Petty projection inequality and the Hölder inequality [10, p. 140] to obtain

$$\left(\omega_n^{n-1}/\omega_n^{n-1}\right)[V(K_1) \cdots V(K_n)]^{(n-1)/n} \leq S_{-1}(K_1, \ldots, K_n),$$

with equality if and only if the $K_i$ are homothetic ellipsoids.

The inequality (6) is a strengthened form of

$$\left(\omega_n^{n-1}/\omega_n^{n-1}\right)[V(K_1) \cdots V(K_n)]^{(n-1)/n} \leq S_1(K_1, \ldots, K_n),$$

with equality if and only if all $K_i$ are balls.

In connection with the last two inequalities, we note that a centrally symmetric body $K$ always has volume greater than that of any other convex body whose brightness function is the same as that of $K$ (see [18 and 24]).

If we combine (5) and (6) we obtain an inequality in the spirit of the concurrent cross-section inequality of Busemann [4] (also see [17]):

$$\frac{1}{n} \int_{S^{n-1}} l_{K_1}(u) \cdots l_{K_n}(u) dS(u) \leq (\omega_n^n/\omega_{n-1}^n)[V(K_1) \cdots V(K_n)]^{1/n},$$

with equality if and only if the $K_i$ are homothetic ellipsoids.

For a convex body $K$ and a direction $u \in S^{n-1}$, let $V_K(u)$ denote the volume of the right cylinder circumscribed about $K$ whose generators are orthogonal to $u$. Clearly $V_K(u) = b_K(u)\sigma_K(u)$. By using the inequalities of Theorems 1 and 2 in conjunction with the Hölder inequality we obtain

$$\frac{2\omega_n^{n-1}}{\omega_n} V(K) \leq \left[ \frac{1}{n \omega_n} \int_{S^{n-1}} V_K(u)^{-1} dS(u) \right]^{-1},$$

with equality if and only if $K$ is a ball.

This is a strengthened form of the inequality

$$\frac{2\omega_n^{n-1}}{\omega_n} V(K) \leq \frac{1}{n \omega_n} \int_{S^{n-1}} V_K(u) dS(u),$$

with equality if and only if $K$ is a ball,

which was obtained by Knothe [11] for $n = 3$ and proved by Chakerian [5] for $n \geq 3$.

The author would like to thank Professors R. Schneider and K. Leichtweiss for several informative observations regarding the Blaschke-Santaló inequality. The author would also like to thank the referee for reference [22].
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