THE JUMP INVERSION THEOREM FOR $Q_{2n+1}$-DEGREES

ILIAS G. KASTANAS

Abstract. Assuming projective determinacy we extend Friedberg's Jump Inversion theorem to $Q_{2n+1}$-degrees, after noticing that it fails for $\Delta_{2n+1}^\mathbf{1}$-degrees.

0. Preliminaries. We list some results from the theory of countable analytical sets and $Q$-theory. For a more extensive account, including proofs, see [2 and 5]. Some familiarity with forcing in the analytical hierarchy is assumed; consult [3 and 4].

Definition 0.1 (PD). $C_{2n+1}$ is the largest countable $\Pi^1_{2n+1}$ set of reals.

Definition 0.2 (PD). $C_{2n+2}$ is the largest countable $\Sigma^1_{2n+2}$ set of reals.

We mention some of their properties: $C_{2n+2}$ is the set of reals that are recursive in some element of $C_{2n+1}$. The set $C_m$ is made up of $\Delta_m^1$-degrees (a $\Delta_m^1$-degree is a set of reals that is an equivalence class for the equivalence relation $\equiv_{\Delta_m^1} \Delta_m^1(\alpha)$, and $\beta \in \Delta_m^1(\alpha)$). The $\Delta_m^1$-degrees in the set $C_m$ are well-ordered by $\equiv_{\Delta_m^1} \Delta_m^1(\beta)$.

Definition 0.3. Given $S \subseteq \omega^\omega$ let $H_{2n+1}(S) = \{ \alpha: \forall \beta \in S (\alpha \in \Delta^1_{2n+1}^\beta(\beta)) \}$; we call it the hull of $S$. If $S$ is a nonempty $\Sigma^1_{2n+1}$ set then $H_{2n+1}(S)$ is called a $\Sigma^1_{2n+1}$-hull. We let $Q_{2n+1}$ be the union of all $\Sigma^1_{2n+1}$-hulls.

We have, assuming PD: The set $Q_{2n+1}$ is $\Pi^1_{2n+1}$-closed. Every $\Sigma^1_{2n+1}$-hull is $\Pi^1_{2n+1}$-bounded (this means that if $R(\alpha, x)$ is $\Pi^1_{2n+1}$ this so is $\exists \alpha \in H_{2n+1}(S) R(\alpha, x)$). The set $Q_{2n+1}$ is the largest $\Sigma^1_{2n+1}$-hull, and the largest $\Pi^1_{2n+1}$-bounded set. Relativizing to an arbitrary real $\beta$ we may define the set $Q_{2n+1}(\beta)$. We define also $\equiv_{Q_{2n+1}} \equiv_{\Delta^1_{2n+1}}$, and $\equiv_{Q_{2n+1}} \equiv_{\Delta^1_{2n+1}}$. This is an equivalence relation, and the equivalence classes are called $Q_{2n+1}$-degrees. The set $C_{2n+1}$ consists of such degrees. The set $Q_{2n+1}$ is the largest initial segment of $C_{2n+1}$ closed under $\equiv_{\Delta_{2n+1}}$; it consists of the $\Delta_{2n+1}$-degrees in $C_{2n+1}$ up to and not including the degree of the first nontrivial (i.e. non-$\Delta_{2n+1}$) $\Pi^1_{2n+1}$ singleton $y^2_{2n+1}$. Relativizing to $\alpha$ we have $y^2_{2n+1}$. If $\alpha \equiv_{Q_{2n+1}} \beta$ then $y^2_{2n+1} \equiv_{\Delta_{2n+1}} y^2_{2n+1}$, and $y^2_{2n+1}$ plays the role of the jump for $Q_{2n+1}$-degrees. The set $Q_{2n+1}$ is closed under the $\Delta_{2n+1}$-jump.

To obtain an ordinal assignment for the $Q_{2n+1}$-degrees we proceed as follows. Definition 0.4.

$$\lambda_{2n+1} = \sup \{ \xi: \xi \text{ is the length of a } \Sigma^1_{2n+1} \text{ wellfounded relation on } \omega^\omega \}$$

$$= \sup \{ \xi: \xi \text{ is the length of a } \Delta^1_{2n+1} \text{ prewellordering of } \omega^\omega \}. $$
Relativizing to $\alpha$ we obtain $\lambda_{2n+1}(\alpha)$. Finally

$$k_{2n+1}(\alpha) = \sup \{ \lambda_{2n+1}(\langle \alpha, \beta \rangle) : \lambda_{2n+1}(\langle \alpha, \beta \rangle) < \lambda_{2n+1}(\gamma_{\alpha}^{2n+1}) \}.$$  

Of course, $\lambda_{2n+1}$ is the ordinal assignment for the $\Delta_{2n+1}^\Sigma$-degrees, e.g. the Spector Criterion holds: $d \leq \Delta_{2n+1}^\Sigma e \Rightarrow [d' \leq \Delta_{2n+1}^\Sigma e \Rightarrow \lambda_{2n+1}(d) < \lambda_{2n+1}(e)]$. Now we have $\lambda_{2n+1}(\alpha) < k_{2n+1}(\alpha) < \lambda_{2n+1}(\gamma_{\alpha}^{2n+1})$, $k_{2n+1}(\alpha)$ is invariant under $\equiv_{Q_{2n+1}}$, $\alpha \leq \lambda_{2n+1}(\beta) \Rightarrow k_{2n+1}(\alpha) \leq k_{2n+1}(\beta)$, and the Spector Criterion is true for $\Sigma_{2n+1}$-degrees: $d \leq Q_{2n+1} e \Rightarrow [d' \leq Q_{2n+1} e \Rightarrow k_{2n+1}(d) < k_{2n+1}(e)]$. Naturally $d'$ is the degree of $\gamma_d^{2n+1}$.

The relation $k_{2n+1}(\alpha) < k_{2n+1}(\beta)$ is $\Sigma_{2n+1}$.

1. Background and definitions. One of the early results in the theory of Turing degrees was the following:

**Friedberg Jump Inversion theorem [1]**. If $b \geq b'$ then there exists an $a$ such that $a' = a \lor b' = b$.

Of course $0$ denotes the degree of the recursive sets, and $'$ denotes the Turing jump operation.

Next, the question was considered in the context of hyperdegrees. Let $0$ denote the hyperdegree of the hyperarithmetical sets and $'$ the hyperjump. Does the above theorem hold? The answer is yes [6]:

**Jump Inversion theorem for $\Delta_1^\Sigma$-degrees.** If $b \geq b'$ then there exists an $a$ such that $a' = a \lor b' = b$.

A natural question now is: does the inversion theorem hold for $\Delta_{2n+1}^\Sigma$-degrees? (We are assuming PD, needless to say). By a well-known argument Determinacy implies that there exists some cone on which inversion holds (a cone, by definition, is $\{a : a \geq b\}$, and $b$ is called the base of the cone). But what is the base of the cone? Is it again $0$? (I.e. the $\Delta_{2n+1}^\Sigma$-jump of the degree of $\Delta_{2n+1}^\Sigma$ sets.) Surprisingly, the answer is no.

**Theorem (Kechris, unpublished) (PD).** If $n \geq 1$, then no real in $C_{2n+2}$ can be a base for a cone of inversion of the $\Delta_{2n+1}^\Sigma$-jump. ("Cone of inversion" of course means that every member of the cone is the $\Delta_{2n+1}^\Sigma$-jump of some $\Delta_{2n+1}^\Sigma$-degree.)

**Proof.** For notational simplicity we let $2n + 1 = 3$. If a member of $C_4$ were abase then it would be recursive in a member of $C_3$, so without loss of generality assume a base $b$ is in $C_3$. Consider the set $C = \{ \alpha : \exists \beta \in Q_3(\alpha) (\beta \in C_3 and \alpha \leq \Delta_3^\Sigma \beta) \}$. It is a subset of $C_4$, and it is $\Pi_3^1$ because the quantification is bounded. So it is countable, and hence a subset of $C_3$. Since $b \in C_3$ everything $\geq b$ in $C_3$ is the $\Delta_3^\Sigma$-jump of a member of $C$, thus a member of $C_3$. However the $\Delta_3^\Sigma$-degrees in $C_3$ are wellordered with successor steps taken by the $\Delta_3^\Sigma$-jump, so that a limit stage of this wellordering gives immediately a contradiction. ($C_3$ is closed under $\equiv_{Q_3}$, hence $\alpha' \in C_3 \Rightarrow \alpha \in C_3$, hence no limit level of $C_3$ is a $\Delta_3^\Sigma$-jump.)

So the inversion theorem is a property of hyperdegrees that fails to generalize to $\Delta_{2n+1}^\Sigma$-degrees, $n \geq 1$. Usually in such cases the validity of the property is restored if instead of $\Delta_{2n+1}^\Sigma$-degrees we work with $Q_{2n+1}$-degrees. Indeed, it is the case that the jump inversion theorem holds for $Q_{2n+1}$-degrees, i.e. the base is again $0$. Moreover we can establish that the $Q_{2n+1}$-jump is never one-to-one.
Jump Inversion theorem for $Q_{2n+1}$-degrees (PD). If $c$ is a $Q_{2n+1}$-degree $\geq 0'$ then there exist $Q_{2n+1}$-degrees $a, b$ such that $a \lor b = a' = b' = c$.

The rest of the paper is devoted to the proof of this theorem.

2. The proof. For notational simplicity we work with $2n + 1 = 3$. First we establish a lemma.

Lemma 2.1. If $0' \not\leq b$ (i.e. $k_3^0 = k_3^b$) then $b' = b \lor 0'$.

Proof. By the Spector Criterion $0' \not\leq b$ iff $k_3^0 = k_3^b$. Now $k_3^0 < k_3^{b \lor 0'}$, so again by the Spector Criterion $b' \leq b \lor 0'$. The opposite inequality is obvious.

Proof of the theorem. The set $\{a: k_3^a = k_3^0 \text{ and } a \not\in Q_3\}$ is $\Sigma_1^0$- and comeager. In fact there is a sequence $D_0, D_1, \ldots$ of dense open sets, $\{D_i\} \in \Delta_3(y_0)$, such that $\cap D_i \subset \{a: k_3^a = k_3^0 \text{ and } a \not\in Q_3\}$. This is implicit in [3]; briefly, comeagerness is characterized by the Banach-Mazur game. Use the Game Formula to unfold it and make it $\Pi^1_2$; then the set of winning strategies is also $\Pi^1_2$, so there is a winning strategy recursive in $y_0$, by the Martin-Solovay basis theorem [5]. This gives the dense open sets.

We describe an inductive construction of reals $a$ and $b$. Set $a_{-1} = b_{-1} = \varnothing$.

Inductive step. Suppose $a_n, b_n$ have been constructed (they are finite sequences of integers). Consider the dense, open set $D_{n+1}$ and extend $a_n$ by a finite segment $s$, least in some fixed enumeration, so that the basic neighborhood defined by $a_n s$ is contained in $D_{n+1}$. Extend $b_n s$ by a finite segment $t$, least again, so that the basic neighborhood defined by $b_n s t$ is contained in $D_{n+1}$. Set now $a_{n+1} = a_n s t \{c(n)\}$, $b_{n+1} = b_n s t \{c(n) + 1\}$.

This completes the inductive step.

Let now $a = \bigcup a_n, b = \bigcup b_n$. Since $a, b \in \cap D_i$ we have by Lemma 2.1 that $a' = a \lor 0', b' = b \lor 0'$. Now $a \lor 0' \geq c$, because using $y_0$ we may trace the construction of $a$ and find all $c(n)$'s. Likewise $b \lor 0' \geq c$. However $a \lor 0' \leq c$, too, because $0' \leq c$ and the construction of $a$ only needs $y_0$ and $c$. The same holds for $b$, and therefore we have $a' = b' = a \lor 0' = b \lor 0' = c$. Finally note that $a \lor b \geq c$, because if both $a$ and $b$ are available then considering the points where they differ $c$ may be obtained. So we have $a' = b' = a \lor b = c$, and $a, b$ cannot have the same degree.

Remark. The real $a, b$ may also be chosen to be of minimal degree by using perfect trees in $Q_3$ instead of finite sequences.

References

4. _______, Forcing with $\Delta$ perfect sets and minimal $\Delta$-degrees, J. Symbolic Logic (to appear).

Department of Mathematics, California State University, Los Angeles, 5151 State University Drive, Los Angeles, California 90032