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SOME RANDOM FIXED POINT THEOREMS FOR CONDENSING OPERATORS

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ABSTRACT. In this paper we obtain several random fixed point theorems including a stochastic generalization of the classical Rothe fixed point theorem. The results herein improve a recent result of Bharucha-Reid and Mukherjea and also some similar results of Itoh.

1. Throughout, let (Ω, Σ) be a measurable space (Σ = sigma algebra) and X a nonempty subset of a Banach space E. In a recent paper **[1]**, Bharucha-Reid and Mukherjea (see also Mukherjea **[6]**) have given sufficient conditions for a stochastic analogue of Schauder's fixed point theorem for a random operator $T: \Omega \times X \to X$. Itoh **[5]** introduced random condensing operators and considerably improved their result (see Lemma 1 below). In this paper, we consider random operators $T: \Omega \times X \to E$ and give sufficient conditions for the existence of a measurable map $\phi: \Omega \to X$ satisfying a Browder-Fan **[4]** type result. As a consequence, a stochastic generalization of the well-known Rothe fixed point theorem is obtained. A random analogue of the Krasnoselskii fixed point theorem for the sum of two operators is given for a Hilbert space.

2. Definitions. For most definitions we refer to Bharucha-Reid **[2]**. A mapping $\phi: \Omega \to E$ is measurable iff $\phi^{-1}(U) \in \Sigma$ for each open subset U of E. The mapping $T: \Omega \times X \to E$ is a random operator iff for each fixed $x \in X$, the mapping $T(\cdot, x): \Omega \to E$ is measurable, and it is continuous if for each $\omega \in \Omega$, the mapping $T(\omega, \cdot): X \to E$ is continuous. A measurable mapping $\phi: \Omega \to E$ is a random fixed point of the random operator $T: \Omega \times X \to E$ iff $T(\omega, \phi(\omega)) = \phi(\omega)$ for each $\omega \in \Omega$.

For a subset A of E, let $\delta(A)$ denote the diameter, $\operatorname{int}(A)$ the interior, $\operatorname{co}(A)$ the convex hull, $\partial(A)$ the boundary and $\gamma(A)$ the measure of noncompactness of the set A. Note that $\gamma(A) = \inf\{\varepsilon > 0: A$ can be covered by a finite number of subsets $A_1, A_2, \ldots, A_n$ of E with $\delta(A_i) \leq \varepsilon$ for each $i\}$. It is easy to show that (i) $\gamma(A) = 0$ iff A is totally bounded and $\gamma(A + B) = \gamma(A)$ if B is totally bounded, (ii) $\gamma(A) \leq \gamma(B)$ if $A \subseteq B$ with $\gamma(\operatorname{co}(A)) = \gamma(A)$. A random operator $T: \Omega \times X \to E$ is called (a) condensing if for each bounded subset $A \subseteq X$ with $\gamma(A) > 0$, $\gamma(T(\omega, A)) < \gamma(A)$ for each fixed $\omega \in \Omega$; (b) contraction if there exists a mapping $\alpha: \Omega \to [0, 1)$ satisfying

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$\|T(\omega, x) - T(\omega, y)\| \leqslant \alpha(\omega)\|x - y\|$ for each $x, y \in X$, $\omega \in \Omega$; (c) compact (bounded) iff for each fixed $\omega \in \Omega$, the set $T(\omega, X)$ is totally bounded (bounded) subset of E.

Recall that if U is a convex neighborhood of the origin in E, then the Minkowski function p of U is defined on E by $p(x) = \inf\{\varepsilon > 0\colon x \in \varepsilon U\}$. It is continuous, subadditive, positively homogeneous and satisfies $\{x\colon p(x) < 1\} \subseteq U \subseteq \{x\colon p(x) \leqslant 1\} \subseteq$ closure of G.

The result below is due to Itoh [**5**, Theorem 2.1] and is basic to our main result.

LEMMA 1. *Let X be a separable closed and convex subset of a Banach space E and $T\colon \Omega \times X \to X$ be a bounded, continuous condensing random operator. Then T has a random fixed point.*

3. The following is a Browder-Fan [**4**] type result.

THEOREM 1. *Let X be a separable, closed and convex subset of a Banach space E with* $\operatorname{int}(X) \neq \varnothing$ *and let $T\colon \Omega \times X \to E$ be a bounded, continuous condensing random operator. Then for each $y \in \operatorname{int}(X)$, there exists a measurable map $\phi = \phi(y)\colon \Omega \to X$ satisfying for each $\omega \in \Omega$,*

$$p(T(\omega, \phi(\omega)) - \phi(\omega)) = \min\{p(T(\omega, \phi(\omega)) - x)\colon x \in X\}, \tag{1}$$

where $p = p(y)$ is the Minkowski function of $(X - y)$. Further, if

$$p(T(\omega, \phi(\omega)) - y) \leqslant 1 \quad \textit{for some } \omega \in \Omega,$$

then $T(\omega, \phi(\omega)) = \phi(\omega)$.

As a consequence of Theorem 1, we have the following stochastic generalization of the well-known Rothe fixed point theorem. Note that for any closed X of E if $x \in \operatorname{int}(X)$ and $y \in E \setminus X$, then there exists [**1**, Lemma 1] a, c with $0 < c < 1$ such that $cx + (1 - c)y \in \partial X$.

THEOREM 2. *Let X be a separable, closed and convex subset of a Banach space E and $T\colon \Omega \times X \to E$ be a bounded, continuous condensing random operator. If $T(\Omega \times \partial X) \subseteq X$, then T has a random fixed point.*

PROOF. If $X = \partial X$, then by Lemma 1, T has a random fixed point. If $\operatorname{int}(X) \neq \varnothing$, then for any $y \in \operatorname{int}(X)$, it follows by Theorem 1 that there is a measurable map $\phi = \phi(y)\colon \Omega \to X$ satisfying (1). To show ϕ is in fact a random fixed point, it suffices to show that $p(T(\omega, \phi(\omega)) - y) \leqslant 1$ for each $\omega \in \Omega$. Suppose for some $\omega \in \Omega$, $p(T(\omega, \phi(\omega)) - y) > 1$. Then $T(\omega, \phi(\omega)) \notin X$. This implies by hypothesis that $\phi(\omega) \in \operatorname{int}(X)$. Consequently, there is a c with $0 < c < 1$, such that $z = c\phi(\omega) + (1 - c)T(\omega, \phi(\omega)) \in \partial X$. Hence by (1),

$$p(T(\omega, \phi(\omega)) - \phi(\omega)) \leqslant p(T(\omega, \phi(\omega)) - z) = cp(T(\omega, \phi(\omega)) - \phi(\omega)).$$

This yields $p(T(\omega, \phi(\omega)) - \phi(\omega)) = 0$. This implies that $p(T(\omega, \phi(\omega)) - y) \leqslant p(\phi(\omega) - y) \leqslant 1$, which contradicts the assumption.

It may be remarked that since a compact random operator is bounded and condensing, Theorem 10 in [**1**] is a special case of Theorem 2.

The lemma below simplifies the proof of Theorem 1.

LEMMA 2. *Let* $f: \Omega \to E$ *and* $l: \Omega \to [0,1]$ *be measurable mappings. Then for any fixed* $y \in E$, *the mapping* $h: \Omega \to E$ *defined by* $h(\omega) = l(\omega)f(\omega) + (1 - l(\omega))y$ *is measurable.*

PROOF. We first show that $k(\omega) = l(\omega) \cdot f(\omega)$ is measurable. To prove this, choose a sequence $\{S_n\}$ of simple functions, $S_n: \Omega \to [0,1]$ such that $S_n \to l$ uniformly. Let $S_n = \sum_{i=1}^{m(n)} \alpha_{in} \chi_{E_{in}}$ where $E_{in} \cap E_{jn} = \varnothing$ if $i \neq J$, $\bigcup_{i=1}^{m(n)} E_{in} = \Omega$, $E_{in} \in \Sigma$ and $\alpha_{in} \in [0,1]$ for each i. Define a sequence $g_n: \Omega \to E$ by $g_n(\omega) = S_n(\omega) \cdot f(\omega)$. Then for each open set U in E

$$g_n^{-1}(U) = \bigcup_{i=1}^{m(n)} \begin{cases} E_{in} \cap f^{-1}(\alpha_{in}^{-1}U) & \text{if } \alpha_{in} \neq 0, \\ E_{in} & \text{if } \alpha_{in} = 0 \text{ and } \theta \in U, \\ \phi & \text{if } \alpha_{in} = 0 \text{ and } \theta \notin U. \end{cases}$$

Thus, g_n is measurable for each n. Since $g_n \to k$, it follows that k is measurable. Now, since $h(\omega) = l(\omega)(f(\omega) - y) + y$ and $f + u$ is measurable for each $u \in E$, it follows that h is measurable.

PROOF OF THEOREM 1. Let y and p be as in Theorem 1. Define a mapping $l: \Omega \times X \to [0,1]$ by

$$l(\omega, x) = \max\{1, p(T(\omega, x) - y)\}^{-1}. \tag{2}$$

Then l is a continuous random operator and $l(\omega, x)p(T(\omega, x) - y) \leqslant 1$ for each (ω, x). Define $g: \Omega \times X \to E$ by

$$g(\omega, x) = l(\omega, x)T(\omega, x) + (1 - l(\omega, x))y. \tag{3}$$

We show that g satisfies the conditions of Lemma 1. Clearly by hypothesis and (3), g is a bounded, continuous random operator and since $p(g(\omega, x) - y) = l(\omega, x)p(T(\omega, x) - y) \leqslant 1$, it follows that $g(\Omega \times X) \subseteq X$. Further, for any bounded set B of X with $\gamma(B) > 0$ and $\omega \in \Omega$, it follows by (3) that

$$\gamma(g(\omega, B)) \leqslant \gamma(\mathrm{co}(T(\omega, x) \cup y)) = \gamma(T(\omega, B)) \leqslant \gamma(B).$$

Thus, g is condensing. Consequently, by Lemma 1, there exists a measurable map $\phi: \Omega \to X$ with $g(\omega, \phi(\omega)) = \phi(\omega)$ for each $\omega \in \Omega$. This implies that for each $\omega \in \Omega$

$$l(\omega, \phi(\omega))T(\omega, \phi(\omega)) + (1 - l(\omega, \phi(\omega)))y = \phi(\omega). \tag{4}$$

Now for any $\omega \in \Omega$, either (a) $p(T(\omega, \phi(\omega)) - y) \leqslant 1$ or (b) $p(T(\omega, \phi(\omega)) - y) > 1$. In case of (a), it follows by (2) $l(\omega, \phi(\omega)) = 1$ and hence by (4) $T(\omega, \phi(\omega)) = \phi(\omega)$. If (b) holds then by (2) $l(\omega, \phi(\omega))p(T(\omega, \phi(\omega)) - y) = 1$ and since for any $x \in X$, $p(x - y) \leqslant 1$, it follows by (4) that for any $x \in X$,

$$p(T(\omega, \phi(\omega)) - \phi(\omega)) = p(T(\omega, \phi(\omega)) - y) - 1 \leqslant p(T(\omega, \phi(\omega)) - x).$$

Since $\phi(\omega) \in X$, the last inequality implies that $p(T(\omega, \phi(\omega)) - \phi(\omega)) = \min\{p(T(\omega, \phi(\omega)) - x) : x \in X\}$. Thus (a) and (b) provide the conclusions of Theorem 1.

If X is a closed and convex subset of a Hilbert space H, then it is well known (see Cheney and Goldstein [3]) that there exists a metric projection $P: H \to X$ such that

P is nonexpansive ($\|Px - Py\| \leqslant \|x - y\|$, $x, y \in H$) and satisfies $\|Px - x\| = \min\{\|z - x\|: z \in X\}$ for each $x \in H$. Using the metric projection, an interesting and also a stronger random version (see [**5**, Theorem 2.4] or [**1**, Theorem 12]) of Krasnoselskii's fixed point theorem can be obtained in H. In fact, we have

THEOREM 3. *Let X be a separable closed and convex subset of a Hilbert space H and $A, B : \Omega \times X \to H$ be random operators such that A is a contraction and B is compact and continuous. Then there exists a measurable map $\phi : \Omega \to X$ such that for each $\omega \in \Omega$,*

$$\|T(\omega, \phi(\omega)) - \phi(\omega)\| = \min\{\|T(\omega, \phi(\omega)) - x\|: x \in X\}$$

where $T = A + B$. If additionally $T(\Omega \times \partial X) \subseteq X$, then ϕ is a random fixed point.

PROOF. Let y be a fixed element of X. Define $g = g(y) : \Omega \times X \to X$ by $g(\omega, x) = P(A(\omega, x) + B(\omega, y))$, where P is the metric of H onto X. Then g is a random contraction operator and hence (see Bharucha-Reid [**2**, p. 109]) there exists a measurable map $\xi = \xi(y) : \Omega \to X$ satisfying for $\omega \in \Omega$, $P(A(\omega, \xi(\omega)) + B(\omega, y)) = \xi(\omega)$. Define $h : \Omega \times X$ by $h(\omega, y) = \xi(y)(\omega)$, $(\omega, y) \in \Omega \times X$. Then h is a random operator and satisfies

$$P(A(\omega, h(\omega, y)) + B(\omega, y)) = h(\omega, y) \tag{5}$$

for each $(\omega, y) \in \Omega \times X$. It follows by (5) that for $\omega \in \Omega$, $x, y \in X$, $\|h(\omega, x) - h(\omega, y)\| \leqslant (1 - \alpha(\omega))^{-1}\|B(\omega, x) - B(\omega, y)\|$, where $\alpha(\omega)$ is the contraction constant (see definition). This implies that h is bounded and continuous. Further, since $\gamma(B(\omega, X)) = 0$, it follows by (5) that $\gamma(h(\omega, X)) \leqslant \gamma(A(\omega, h(\omega, X))) \leqslant \alpha(\omega)\gamma(h(\omega, X))$ and consequently $\gamma(h(\omega, X)) = 0$ for each $\omega \in \Omega$. This implies that h is condensing. Thus by Lemma 1, there is a measurable map $\phi : \Omega \to X$ with $h(\omega, \phi(\omega)) = \phi(\omega)$ for each $\omega \in \Omega$. Thus, $P(A(\omega, \phi(\omega)) + B(\omega, \phi(\omega))) = \phi(\omega)$ and in view of the definition of P, we have

$$\|T(\omega, \phi(\omega)) - \phi(\omega)\| = \min\{\|T(\omega, \phi(\omega)) - x\|: x \in X\}.$$

The proof of the last part is contained in the proof of Theorem 2. In fact, if for some $\omega \in \Omega$, $T(\omega, \phi(\omega)) \neq \phi(\omega)$, then by the above conclusion, $\phi(\omega) \in \operatorname{int}(X)$ and $T(\omega, \phi(\omega)) \in H \setminus X$. Consequently, there is a c, $0 < c < 1$ with

$$(1 - c)\|T(\omega, \phi(\omega)) - \phi(\omega)\| = 0,$$

which contracts the assumption.

It may be remarked that the random analogue of Krasnoselskii's theorem [**5**, Theorem 2.4], or [**1**, Theorem 12] in E requires the additional hypothesis that $A(\omega, x) + B(\omega, y) \in X$ for each $\omega \in \Omega$ and $x, y \in X$. Thus, Theorem 3 in H is an improvement over E.

References

1. A. T. Bharucha-Reid, *Fixed point theorems in probabilistic analysis*, Bull. Amer. Math. Soc. **82** (1976), 641–645.

2. ______, *Random integral equations*, Academic Press, New York, 1972.

3. W. E. Cheney and A. A. Goldstein, *Proximity maps for convex sets*, Proc. Amer. Math. Soc. **10** (1959), 448–450.

4. Ky Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z. **112** (1969), 230–240.

5. S. Itoh, *Random fixed point theorems with an application to random differential equations in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 261–273.

6. A. Mukherjea, *Transformations aléatoires séparables*, *Théorème du point fixe aléatoire*, C. R. Acad. Sci. Paris Sér. A-B **263** (1966), 393–395.

7. C. H. Su and V. M. Sehgal, *Some fixed point theorems for nonexpansive mappings in locally convex spaces*, Boll. Un. Mat. Ital. (4) **10** (1974), 598–601.

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