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AN ELEMENTARY APPROACH TO GENERIC PROPERTIES OF PLANE CURVES

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ABSTRACT. A direct and geometrical approach to the proof of generic properties of smooth submanifolds of Euclidean space is illustrated in the simplest case of plane curves.

Arguments for establishing generic properties of smooth submanifolds of Euclidean space usually involve parametrised families of maps into jet and multijet spaces, and transversality to stratifications of those spaces by orbits under certain group actions. In this note we show how to obtain some of these results by a more direct approach to the geometry. It is still necessary to make standard topological arguments in order to pass from a local transversality statement to the required genericity result; these arguments are not always spelt out in what follows. We shall concentrate on the geometry, explaining the tool we use (the Monge-Taylor map) and the proof of the transversality statement we require. (The Monge-Taylor map was originally used by one of us in a situation where the standard techniques proved inadequate [**1**].)

We keep to the simplest case of plane curves in what follows, but the same ideas work in general.

1. The Monge-Taylor map. (Compare [**1**].) Given a smooth embedding $e\colon S \to \mathbf{R}^2$, where S is the unit circle, and a point $p = e(t)$, let the local equation of $e(S)$ referred to axes along the oriented tangent and normal at p be $\eta = f(\xi)$. (This is the equation of $e(S)$ at p in "Monge normal form".) Then define a smooth map $\gamma = \gamma_e\colon S \to V_k$, where V_k ($k \geqslant 2$) is the vector space of polynomials of degree $\geqslant 2$ and $\leqslant k$ by $\gamma(t) =$ Taylor expansion of f at 0 up to and including the term in ξ^k. (For curves and surfaces in $\mathbf{R}^3$ the definition of γ needs some care (see [**1**]).)

Let U_1 be the vector space of polynomial mappings $\mathbf{R}^2 \to \mathbf{R}^2$, both of whose components have degree $\leqslant k$, and let U be a sufficiently small open neighborhood of the identity map id so that $\psi \circ e$ is an embedding for all $\psi \in U$. Then it is not hard to show that the family $\Gamma = \Gamma_e\colon S \times U \to P_k$, defined by $\Gamma(t, \psi) = \gamma_{\psi \circ e}(t)$ is a submersion at every point (t, id). Standard arguments using the compactness of S and Thom's basic transversality lemma [**2**, p. 53] now show that for any submanifold Q of V_k, the embeddings e for which γ_e is transverse to Q are everywhere dense. The

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same goes for transversality to any finite collection of submanifolds $Q_1, \ldots, Q_n$. If the Q_i are closed then we can also deduce that the set of embeddings giving transversality is open in the space of all embeddings.

Various submanifolds have immediate geometric significance. For example, writing $(a_2, \ldots, a_k)$ as coordinates in V_k:

(i) $k = 2$, $a_2 = 0$ gives the inflexions of $e(S)$.

(ii) $k = 3$, $a_3 = 0$ gives the vertices of $e(S)$.

(iii) $k = 3$, $a_2 = a_3 = 0$ gives the higher inflexions of $e(S)$.

(iv) $k = 4$, $a_3 = a_4 - a_2^3 = 0$ gives the higher vertices of $e(S)$.

So we can deduce that for an open dense set of embeddings e the curve $e(S)$ will have finitely many ordinary inflexions and vertices and no higher ones.

2. Quasi-global (or multilocal) questions. The techniques of §1 allow us to prove results about generic curves which are of a purely local nature. Assertions such as "generic curves have no tritangents" cannot be proved using the map γ only. One natural way of attacking such quasi-global questions is as follows. The map γ determines the infinitesimal information concerning the curve at each of its points. To describe the curve locally in a complete way (at least up to order k), we also need the point $e(t)$ in question, and the normal at $e(t)$. Thus we consider the map

$$\delta = \delta_e \colon S \to \mathbf{R}^2 \times S \times V_k = Y_k$$

given by $\delta(t) = (e(t), n(t), \gamma(t))$, where $n(t)$ is the outward unit normal to $e(S)$ at $e(t)$. We now need to find a good family of deformations of the map δ. If E denotes the Euclidean group of rigid motions of $\mathbf{R}^2$ and U is as before, consider the map $\Delta = \Delta_e \colon S \times E \times U \to Y_k$ defined by $\Delta(t, \theta, \psi) = \delta_{\theta \circ \psi \circ e}(t)$.

We claim this is a submersion. The E factor takes care of tangent vectors to $\mathbf{R}^2 \times S$, and gives no vectors in V_k, since γ is invariant under rigid motions of $\psi \circ e(S)$. On the other hand, the U factor as before gives the tangent vectors to V_k (the θ term being irrelevant again; note it is better to use $\theta \circ \psi \circ e$ than $\psi \circ \theta \circ e$ since the latter may not embed S for θ sufficiently far from the identity). Of course, we really wish to study multipoint versions of Δ, so we consider $\Delta^{(r)} \colon S^{(r)} \times (E \times U)^r \to Y_k^r$, where $S^{(r)}$ denotes r-tuples of pairwise distinct points of S, and $\Delta^{(r)}$ is the restriction of the r-fold product of Δ. This is of course a submersion. We now apply the standard arguments [**3**, §2] to deduce that for any submanifold $W \subset Y_k^r$ the set of embeddings $e \colon S \to \mathbf{R}^2$ giving $\delta_e^{(r)} \colon S^{(r)} \to Y_k^r$ transverse to W is residual. As a typical application we prove that for a residual set of embeddings $e \colon S \to \mathbf{R}^2$ there are no tritangents. To do this we can consider $\delta_e^{(3)} \colon S^{(3)} \to (\mathbf{R}^2 \times S)^3$ (no Y_k factor necessary) and let W be the points of $(\mathbf{R}^2 \times S)^3$ giving the same tangent. Then W is easily checked to be a closed submanifold of codimension 4, and so as a generic embedding $e \colon S \to \mathbf{R}^2$ yields a map $\delta_e^{(3)}$ transverse to W, it must miss W, so $e(S)$ has no triple tangents.

Here is a summary of the topological argument mentioned above. We cover $S^{(r)}$ with countably many compact sets each of the form $K = K_1 \times \cdots \times K_r$ where the K_i are disjoint, and write W as the infinite ascending union of compact submanifolds $W = \bigcup_{j=1}^{\infty} W_j$. Clearly, $\delta_e^{(r)}$ is tranverse to W if and only if, for each K and j, $\delta_e^{(r)}$ is

transverse to W_j at each point of K. This is an open condition, and it remains to prove density. Choose a smooth partition of unity $1 = \sum_1^r \lambda_i$ with $\lambda_i \geqslant 0$ and $\lambda_i = 1$ on K_i. We now consider the map $\tilde{e}$: $S \times (E \times U)^r \to \mathbf{R}^2$ defined by

$$\tilde{e}(t, (\theta_1, \psi_1), \ldots, (\theta_r, \psi_r))$$
$$= \left(\sum_{i=1}^{r} (\mathrm{id} - \lambda_i(t)(\mathrm{id} - \theta_i)) \right) \circ \left(\sum_{i=1}^{r} (\mathrm{id} - \lambda_i(t)(\mathrm{id} - \psi_1)) \right) \circ e(t).$$

For $t \in K_i$ this reduces to $\theta_i \circ \psi_i \circ e(t)$. It follows that $\tilde{e}$ induces a map $S^{(r)} \times (E \times U)^r \to Y_k^r$ which is a submersion on $K \times (E \times U)^r$ since it coincides there with $\Delta^{(r)}$ and the result now follows from Thom's transversality lemma. Note that $\tilde{e}$ does not need to be a family of embeddings of S, for we are proving density and the set of embeddings of S in $\mathbf{R}^2$ is open in the space of maps.

In the case of tritangents, this proof shows that triple tangencies can be eliminated by rigidly moving the corresponding pieces of curve (in different directions) and smoothing off.

In a similar way, one can show that, for a residual set of embeddings e, there are no bitangents where one contact is inflexional.

3. Openness and finiteness. One problem with arguments using the noncompact $S^{(r)}$ is that one cannot deduce from them whether quasi-global properties are *open*; the best that one can manage is *residual*. (The same problem arises when more sophisticated techniques involving multijet spaces are used.) Here we illustrate how one can attack the questions of finiteness and openness by considering the tangent singularities of a curve.

First, note that for an open, dense set of embeddings e: $S \to \mathbf{R}^2$, the resulting curve has *finitely many* ordinary inflexions and *no* higher inflexions. These follow, for example, from the technique of §1. The inflexions are inverse images under γ of a closed submanifold of codimension 1 (see (i) in §1). For generic e, γ is transverse to this submanifold and has *compact* domain, from which finiteness follows.

It is not, however, clear that we can deduce that there are finitely many double tangents, since the domain of the relevant map $\delta_e^{(2)}$ is not compact. Consider the *dual* of the curve given by e. It has ordinary cusps corresponding to the ordinary inflexions of the curve, and self-crossings corresponding to double tangents. Moreover the two branches of the dual through such a self-crossing will have distinct tangents (otherwise, dualizing again, the points of contact of the double tangent would have coincided, which cannot happen for an embedded curve). Thus the dual of a generic curve is smooth except for transverse self-crossings and finitely many ordinary cusps. Suppose we have a sequence of bitangents, tangent at p_n and q_n. Using the compactness of $e(S)$ we can suppose $p_n \to p$ and $q_n \to q$. If $p \neq q$ then $e(S)$ has the same tangent at p and q, and since the dual has a corresponding pair of transverse branches this self-intersection of the dual is isolated, contradicting our assumption. If $p = q$ then the dual of some neighborhood of p is smooth or a cusp, again with no self-intersections. Hence there can be only finitely many bitangents for an open dense set of embeddings.

One can also see, using the dual, that embeddings e for which $e(S)$ has only finitely many ordinary inflexions, no higher inflexions and no tritangents are open, as well as dense, in the space of all embeddings.

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