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ATRIODIC ACYCLIC CONTINUA AND CLASS W

JAMES F. DAVIS

ABSTRACT. A continuum M is in class W provided that for each continuum Y and mapping f of Y onto M, each subcontinuum of M is the image under f of some subcontinuum of Y. It is shown that atriodic continua with trivial first Čech cohomology are in class W.

Introduction. In 1972 Lelek introduced a class of continua known as class W. A continuum belongs to class W provided that each mapping from a continuum onto it is weakly confluent (for definition, see below). Arcs are in class W as are all chainable continua [**10**]. Grispolakis and Tymchatyn have investigated class W from the point of view of hyperspaces [**4, 5, 6**]. In [**6**] they proved that atriodic tree-like continua are in class W, answering a question raised by Ingram in [**8**]. They also proved in [**6**] that the Case-Chamberlin continuum (see [**1**]) is in class W. The Case-Chamberlin continuum is an example of a continuum M, such that every mapping of M to the circle is inessential (a property which is equivalent to M being acyclic in the first Čech cohomology group with integer coefficients) but which admits an essential mapping to the figure eight, and thus the continuum is not tree-like. The Case-Chamberlin continuum is also atriodic.

These two theorems of Grispolakis and Tymchatyn thus invite the conjecture that atriodic acyclic continua are in class W. In this paper we show that this is true. We do so without appealing to hyperspace concepts.

One might also conjecture that atriodic unicoherent continua are in class W. We exhibit an example to show that this is not true.

Definitions. All spaces considered in this paper are metric, and by a *continuum* we mean a compact connected metric space. A *mapping* is a continuous function.

Recently Ingram has introduced terminology in [**7**] which is very useful in discussing confluence and weak confluence, and which we now adopt. Suppose M is a continuum, H is a subcontinuum of M and f is a mapping of a continuum onto M. We say that f *is confluent with respect to* H (respectively, *weakly confluent with respect to* H) provided that for each (resp., some) component, L, of $f^{-1}(H)$, $f(L) = H$. We say that f is *confluent* (resp., *weakly confluent*) provided f is confluent (resp., weakly confluent) with respect to each subcontinuum of M.

The continuum M *is in class* W provided that for each mapping f of a continuum onto M, f is weakly confluent.

A continuum is a *triod* provided it contains a subcontinuum whose complement has at least three components. A continuum is *atriodic* provided it contains no

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triod. The continuum M is *unicoherent* provided that if A and B are subcontinua of M such that $M = A \cup B$ then $A \cap B$ is connected.

Denote the unit circle in the complex plane by S^1. Denote the first Čech cohomology group of a space M, with integer coefficients, by $\check{H}^1(M;Z)$. As was mentioned previously, the condition $\check{H}^1(M;Z) = \{0\}$ is equivalent to M having the property that each mapping $f\colon M \to S^1$ is inessential.

Preliminaries. We now marshal certain facts which we will need for the proof of the principal result.

LEMMA 1. *If f is a mapping of a continuum onto the atriodic continuum M then f is confluent with respect to every nonunicoherent proper subcontinuum of M.*

PROOF. This follows immediately from Theorem 4 of [**7**] and Theorem 1.5 of [**2**].

LEMMA 2. *Suppose f is a mapping of the continuum Y onto the continuum M, C is a subcontinuum of M irreducible from p to q, and f is not weakly confluent with respect to C. Let*

$$A = \bigcup\{K\colon K \textit{ is a component of } f^{-1}(C) \textit{ and } p \in f(K)\},$$

and

$$B = \bigcup\{K\colon K \textit{ is a component of } f^{-1}(C) \textit{ and } q \in f(K)\}.$$

Then A and B are mutually exclusive closed subsets of Y.

PROOF. Suppose that x_0 is a limit point of A. Let $x_1, x_2, x_3, \ldots$ be a sequence of points of A with limit x_0. For $i = 1, 2, 3, \ldots$ let K_i be the component of $f^{-1}(C)$ which contains x_i, and let z_i be a point of K_i such that $f(z_i) = p$. By taking subsequences and renaming, if necessary, we may assume that the sequence $K_1, K_2, K_3, \ldots$ has a sequential limiting set, K_0, and that the sequence $z_1, z_2, z_3, \ldots$ has a limit, z_0, in K_0. Since f is continuous, $f(z_0) = p$. Then x_0 is in K_0, K_0 is a continuum, and p is in $f(K_0)$. Thus x_0 is in A, and hence A is closed. Likewise, B is closed.

LEMMA 3. *Suppose X and Y are compact spaces, f is a mapping of X onto Y, C is a closed subset of Y, and U is an open subset of X such that $f^{-1}(C) \subset U$. Then there is an open subset V of Y such that $C \subset V$ and $f^{-1}(V) \subset U$.*

PROOF. Let $I = X - U$. Since I is closed and X is compact, I is compact. Hence, $f(I)$ is compact and therefore closed. Let $V = Y - f(I)$. Then $C \subset V$ and $f^{-1}(V) \subset U$.

LEMMA 4. *Suppose X and Y are continua and f is a mapping of X onto Y.*

(1) *If $C_1, C_2, C_3, \ldots$ is a sequence of proper subcontinua of Y with sequential limiting set C_0 and f is weakly confluent with respect to each C_i, then f is weakly confluent with respect to C_0.*
(2) *If G is a monotonic collection of proper subcontinua of Y whose intersection is C_0 and if f is weakly confluent with respect to each member of G, then f is weakly confluent with respect to C_0.*

PROOF. To prove (1), for each positive integer n, let K_n be a component of $f^{-1}(C_n)$ such that $f(K_n) = C_n$. There is a subsequence $K_{i_1}, K_{i_2}, K_{i_3}, \ldots$ of $K_1, K_2, K_3, \ldots$ which has a sequential limiting set K_0 which is a continuum [**9**, Theorems 58, 59, pp. 23–24]. From the continuity of F it follows that $f(K_0) = C_0$. Letting L be the component of $f^{-1}(C_0)$ containing K_0, we have $f(L) = C_0$.

Part (2) follows by choosing a cofinal (with respect to containment) sequence $C_1, C_2, C_3, \ldots$ in G. Then $C_1, C_2, C_3, \ldots$ has sequential limiting set C_0 and the conclusion follows from (1).

Principal result. We now show that the conjecture mentioned earlier is true.

THEOREM. *If M is an atriodic continuum and $\check{H}^1(M; Z) = \{0\}$, then M is in class W.*

PROOF. Suppose that M is an atriodic continuum, $\check{H}^1(M; Z) = \{0\}$, X is a continuum, and f is a mapping of X onto M.

Suppose that f is not weakly confluent. Then there is a subcontinuum C of M such that no component of $f^{-1}(C)$ is mapped by f onto C. By Lemma 1, C is unicoherent and, since M is atriodic, by [**11**, Theorem 3.2, p. 456] there are points p and q of C between which C is irreducible. Let

$$A = \bigcup\{K : K \text{ is a component of } f^{-1}(C) \text{ and } p \in f(K)\},$$

and

$$B = \bigcup\{K : K \text{ is a component of } f^{-1}(C) \text{ and } q \in f(K)\}.$$

By Lemma 2, A and B are mutually exclusive closed subsets of X. By [**9**, Theorem 44, p. 15], $f^{-1}(C)$ is the union of two mutually exclusive closed subsets of X, one containing A, the other containing B. Thus there exist mutually exclusive open subsets of X, U_A and U_B, such that $A \subset U_A$, $B \subset U_B$ and $f^{-1}(C) \subset U_A \cup U_B$.

There exists an open subset V_0 of M such that $C \subset V_0$ and which has the property that if J is a continuum, $J \subset \overline{V}_0$, and $C \subset J$, then f is not weakly confluent with respect to J. Otherwise one could choose a sequence of continua with sequential limiting set C such that f is weakly confluent with respect to each term of the sequence, a contradiction to Lemma 4. Suppose $V \subset V_0$ is an open set and $C \subset V$. Then

(1) if J is a continuum, $J \subset \overline{V}$, and $C \subset J$, then f is not weakly confluent with respect to J, and (from Lemma 1) J is unicoherent.

From Lemma 3 it follows that there is an open subset V' of M, $C \subset V'$ such that $f^{-1}(V') \subset U_A \cup U_B$. By renaming the sets $V \cap V'$, $f^{-1}(V \cap V') \cap U_A$ and $f^{-1}(V \cap V') \cap U_B$, we may assume that $f^{-1}(V) = U_A \cup U_B$. Let

$$\mathcal{A} = \{T : T \text{ is the closure of a component of } U_A \text{ which intersects } A\}$$

and

$$\mathcal{B} = \{T : T \text{ is the closure of a component of } U_B \text{ which intersects } B\}.$$

Let $\alpha = \overline{\bigcup\{f(T) : T \in \mathcal{A}\}}$ and $\beta = \overline{\bigcup\{f(T) : T \in \mathcal{B}\}}$. Then $\alpha \subset \overline{V}$, $\beta \subset \overline{V}$, $p \in \alpha$, $q \notin \alpha$, $q \in \beta$ and $p \notin \beta$. Thus α and β both intersect C, and $\alpha \cap C$ and $\beta \cap C$ are continua (otherwise $\alpha \cup C$ or $\beta \cup C$ would not be unicoherent, a violation of (1)).

We will now show that $(\alpha \cup C) \cap (\beta \cup C) = C$. Now

$$(\alpha \cup C) \cap (\beta \cup C) = (\alpha \cap \beta) \cup (\alpha \cap C) \cup (C \cap \beta) \cup (C \cap C) = (\alpha \cap \beta) \cup C,$$

so if α and β are mutually exclusive, the assertion is clear. Suppose that α and β intersect. Then we observe the following:

(a) $\alpha \cap \beta$ intersects C, otherwise $\alpha \cup \beta \cup C$ would fail to be unicoherent, a contradiction to (1).

(b) Thus $\alpha \cap C$ and $\beta \cap C$ have a point in common, and thus $(\alpha \cap C) \cup (\beta \cap C)$ is a continuum.

(c) Since p and q are in $(\alpha \cap C) \cup (\beta \cap C)$, and C is irreducible from p to q, $(\alpha \cap C) \cup (\beta \cap C) = C$.

(d) Then $C \subset \alpha \cup \beta$, and, therefore, $\alpha \cap \beta$ is a continuum, otherwise $\alpha \cup \beta$ would not be unicoherent, violating (1), and $(\alpha \cap C) \cap (\beta \cap C)$ is a continuum, otherwise C would not be unicoherent, again a violation of (1).

(e) Finally, $\alpha \cap \beta \subset C$, otherwise $\alpha \cap \beta$, $\alpha \cap C$ and $\beta \cap C$ would form a triod in M. Therefore, $(\alpha \cup C) \cap (\beta \cup C) = C$.

From [**3**, Proposition 4, p. 25] it follows that

(2) if J is a continuum intersecting $\alpha \backslash C$ and $\beta \backslash C$ then $C \subset J$.

Let $H_\alpha = \alpha \cap \operatorname{bdry}(V)$ and $H_\beta = \beta \cap \operatorname{bdry}(V)$. Since $\alpha \cap \beta \subset C \subset V$, H_α and H_β are mutually exclusive. Moreover, no component of $\operatorname{bdry}(V)$ intersects both H_α and H_β, for otherwise, denoting such a component by T, $\alpha \cup C \cup \beta \cup T$ would be a nonunicoherent continuum contained in $\overline{V}$ which contains C, contrary to (1). Hence, by [**9**, Theorem 44, p. 15], $\operatorname{bdry}(V)$ is the union of two mutually exclusive closed sets K_α and K_β, $H_\alpha \subset K_\alpha$, and $H_\beta \subset K_\beta$. Let W_α and W_β be open subsets of M such that $K_\alpha \subset W_\alpha$, $K_\beta \subset W_\beta$, $\overline{W}_\alpha$ and $C \cup \beta \cup \overline{W}_\beta$ are mutually exclusive and $\overline{W}_\beta$ and $C \cup \alpha \cup \overline{W}_\alpha$ are mutually exclusive. This is illustrated in Figure 1.

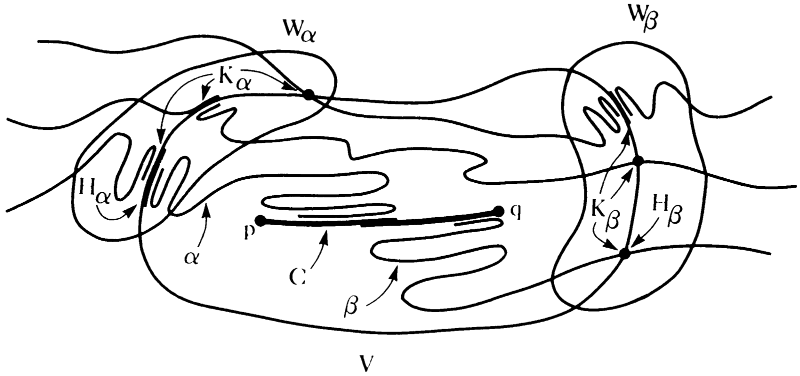

FIGURE 1

Let $J = V \backslash (W_\alpha \cup W_\beta)$. Note that $C \subset J$. The component, L, of J which contains C must be a subset of $\alpha \cup C \cup \beta$, otherwise $L \cup (\alpha \cup C) \cup (\beta \cup C)$ would be a triod.

Suppose that G is an open set containing L. Then there is an open set G' containing L, $G' \subset G \cap V$, such that no point of J is on the boundary of G' [**9**, Theorem 52, p. 21]. Then $J' = J \cap G'$ is closed and L is one of its components.

Let α_0 be the component of $\alpha \cap J$ which intersects C. Then, by [**9**, Theorem 50, p. 18], α_0 contains a point of $\overline{W}_\alpha$. Since $\alpha_0 \subset L \subset J'$, J' intersects $\overline{W}_\alpha$. By a similar argument, J' intersects $\overline{W}_\beta$. Let $F_\alpha = J' \cap \overline{W}_\alpha$ and $F_\beta = J' \cap \overline{W}_\beta$. Then F_α and F_β are mutually exclusive.

Let $\theta_1 : J' \to [0, \pi]$ be a mapping such that $\theta_1^{-1}(0) = F_\alpha$ and $\theta_1^{-1}(\pi) = F_\beta$. Let $\theta_2 : \overline{M \backslash J'} \to [\pi, 2\pi]$ be a mapping such that $\theta_2^{-1}(\pi) = F_\beta$ and $\theta_2^{-1}(2\pi) = F_\alpha$. Define

$g\colon M \to S^1$ by

$$g(x) = \begin{cases} e^{i\theta_1(x)}, & x \in J', \\ e^{i\theta_2(x)}, & x \in \overline{M\backslash J'}. \end{cases}$$

Since $(\overline{M\backslash J'}) \cap J' = F_\alpha \cup F_\beta$, $\theta_2 = \theta_1$ on F_β, $\theta_2 = \theta_1 + 2\pi$ on F_α, and $F_\alpha \cup F_\beta$ is closed, it follows that g is continuous.

Since $\check{H}^1(M; Z) = \{0\}$, each mapping from M to S^1 is inessential. Thus there is a mapping $\phi\colon M \to R$ such that $g(x) = e^{i\phi(x)}$ for each x in M. Thus

$$g(J') = \{e^{it} : t \in [0, \pi]\} \subset \bigcup_{k=-\infty}^{\infty} \phi^{-1}[2\pi k, 2\pi k + \pi],$$

$$g(\overline{M\backslash J'}) = \{e^{it} : t \in [\pi, 2\pi]\} \subset \bigcup_{k=-\infty}^{\infty} \phi^{-1}[2\pi k + \pi, 2\pi k + 2\pi],$$

$$\phi(F_\alpha) = \{2\pi k : -\infty \leq k \leq \infty\} \cap \phi(M),$$

$$\phi(F_\beta) = \{2\pi k + \pi : -\infty \leq k \leq \infty\} \cap \phi(M),$$

$$g(L) = \{e^{it} : 0 \leq t \leq \pi\}.$$

So $\phi(L) = [2\pi k, 2\pi k + \pi]$ for some k. Since $\phi(M) = \phi \circ f(X)$ is an interval, it is in class W and hence $\phi \circ f$ is weakly confluent with respect to $\phi(L)$. Thus there is a component I of $(\phi \circ f)^{-1}(\phi(L))$ such that $\phi \circ f(I) = \phi(L)$. Now $f(I)$ is a continuum and $\phi(f(I)) = [2\pi k, 2\pi k + \pi]$ so

(i) $f(I) \subset J' \subset G$ and

(ii) $f(I)$ intersects both W_α and W_β.

Let $G_1, G_2, G_3, \ldots$ be a sequence of open subsets of M such that $\bigcap_{n>0} G_n = L$. Then for each positive integer n, there exists a continuum $I_n \subset X$ such that

(iii) $f(I_n) \subset G_n$ and

(iv) $f(I_n)$ intersects both W_α and W_β.

Clearly f is weakly confluent with respect to $f(I_n)$ for each n. There is a subsequence $I_{n_1}, I_{n_2}, I_{n_3}, \ldots$ of $I_1, I_2, I_3, \ldots$ such that $f(I_{n_1}), f(I_{n_2}), f(I_{n_3}), \ldots$ has a sequential limiting set L'. Then L' is a continuum which intersects both W_α and W_β, thus L' intersects both $\alpha\backslash C$ and $\beta\backslash C$. From Lemma 4, f is weakly confluent with respect to L'. From (2) it follows that $C \subset L'$, and since $L' \subset \overline{V}$ we have a contradiction to (1). This proves the theorem.

The two results of Grispolakis and Tymchatyn which were mentioned in the Introduction follow as corollaries of this theorem.

COROLLARY 1 [**6**]. *Atriodic tree-like continua are in class W.*

COROLLARY 2 [**6**]. *The Case-Chamberlin continuum is in class W.*

EXAMPLE. To see that atriodic unicoherent continua are not necessarily in class W, let Y be the 2-endpoint Knaster indecomposable continuum which is indicated in Figure 2 (there are actually $c = 2^{\omega_0}$ such 2-endpoint continua [**12**], any one of which will do for this example). Let p and q denote these endpoints. Let p' be another point on the composant of Y determined by p and let q' be another point on the composant of Y determined by q (p and q of course lie on different composants). Let X be the continuum produced by identifying the points p and q, $X = Y/\{p, q\}$, and let f be the quotient map (see Figure 2). Then X is unicoherent (because it is indecomposable) and it is atriodic. Let $[p,p']$ denote the arc in Y which is irreducible from p to p' and let $[q,q']$ denote the arc in Y which is irreducible from q to q'. Then $C = f([p,p']) \cup f([q,q'])$ is a continuum. However $f^{-1}(C) = [p,p'] \cup [q,q']$, its components are precisely $[p,p']$ and $[q,q']$, and neither is mapped by f onto C. Thus X is not in class W.

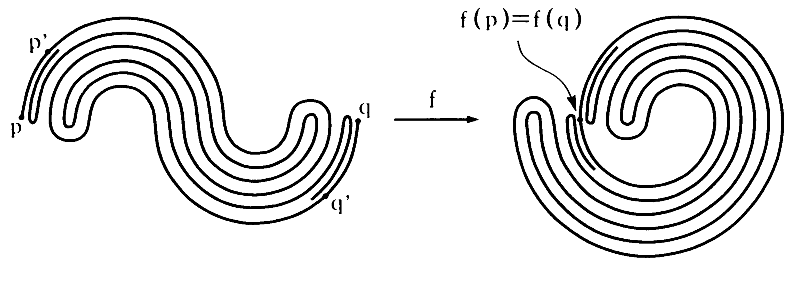

FIGURE 2

We should note that there are atriodic continua in class W which are not acyclic. A circle with a ray spiraling to it is such a continuum.

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TEXACO BELLAIRE RESEARCH LABORATORY, BELLAIRE, TEXAS 77401