IMMERSIONS OF HIGHLY CONNECTED MANIFOLDS

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Abstract. If $[M^n \rightarrow \mathbb{R}^{2n-k}]$ denotes the set of regular homotopy classes of immersions, $M^n$ a $k$-connected compact manifold, we show by a direct geometric construction the correspondence $[M^n \rightarrow \mathbb{R}^{2n-k}] ightarrow [S^n \rightarrow \mathbb{R}^{2n-k}]$ for $0 \leq 2k \leq n-2$.

A bijection $I[M^n \rightarrow \mathbb{R}^{2n-k}] \rightarrow [S^n \rightarrow \mathbb{R}^{2n-k}]$ can be obtained as follows: Any immersion $g: M^n \rightarrow \mathbb{R}^{2n-k}$ is regularly homotopic to an immersion $\tilde{g}$ with $\tilde{g}|M \setminus U$ an embedding, where $U$ is a suitable coordinate neighborhood of the basepoint. Set $I([g]) = \text{the immersion of } S^n \text{ determined by } \tilde{g}|U$. That $I$ is a bijection follows from [2, 3]. This paper gives a simple, direct geometric construction of the map $\tilde{g}$.

Proposition. $M^n$ a compact, closed, $k$-connected manifold. If $0 \leq 2k \leq n-2$, there is a bijection $I: [M^n \rightarrow \mathbb{R}^{2n-k}] \rightarrow \Pi_n(V_n(\mathbb{R}^{2n-k})).$

Remark. The correspondence $[S^n \rightarrow \mathbb{R}^{2n-k}] \rightarrow \Pi_n(V_n(\mathbb{R}^{2n-k}))$ is a classical result of S. Smale [7]. Immersions of spheres are classified by their double points [4, 6]; this result extends this double point classification to highly connected manifolds.

For the proof, we first recall some well-known facts. By [2] a $k$-connected compact closed manifold $M^n$ can be embedded in $\mathbb{R}^{2n-k}$ if $2k \leq n-2$. Moreover, two embeddings are regularly homotopic on $M \setminus \{x_0\}$. These results, together with [7], permit the construction of a group action $[S^n \rightarrow \mathbb{R}^{2n-k}] \times [M^n \rightarrow \mathbb{R}^{2n-k}] \rightarrow [M^n \rightarrow \mathbb{R}^{2n-k}]$ which is seen to be free and transitive, hence the result. Let $g: M^n \rightarrow \mathbb{R}^{2n-k}$ represent a class $[g] \in [M^n \rightarrow \mathbb{R}^{2n-k}]$, and $f: S^n \rightarrow \mathbb{R}^{2n-k}$ represent an immersion with Smale invariant $c(f) \in \Pi_n(V_n(\mathbb{R}^{2n-k}))$. The connected sum of $g$ and $f$ yields an immersion $f \# g: M^n \rightarrow \mathbb{R}^{2n-k}$, and we define $[S^n \rightarrow \mathbb{R}^{2n-k}] \times [M^n \rightarrow \mathbb{R}^{2n-k}] \rightarrow [M^n \rightarrow \mathbb{R}^{2n-k}]$

$$([f],[g]) \mapsto [f \# g].$$

It is obvious that in this way one obtains a group action. If $g$ is as above, we may deform $g$ by a regular homotopy such that $g \times g: M \times M \setminus \Delta_M \rightarrow \mathbb{R}^{2n-k} \times \mathbb{R}^{2n-k}$ is transverse to the diagonal $\Delta \subset \mathbb{R}^{2n-k} \times \mathbb{R}^{2n-k}$.

The set $M_g = \{x \in M| \exists y \in M, x \neq y, g(x) = g(y)\}$ of double points of $g$ is a $k$-dimensional submanifold if $2k \leq n-1$. The embedding $i: M_g \hookrightarrow M$ is null.
homotopic, so we may choose a homotopy $H: M_g \times I \to M$ from $i$ to the trivial map $M_g \to x_0 \in M$. For a suitable coordinate neighbourhood $U \subset M^n$ of $x_0$ and a chart $k: (D^n, 0) \to (U, x_0)$ we can find $\epsilon > 0$ such that $H_t(M_g) \subset U$ for $t \in [1-\epsilon, 1]$. For $2k \leq n-2$ we can approximate $H$ on $[0, 1-\epsilon/2]$ by $\overline{H}$, such that $\overline{H}_t: M_g \to M$ is an embedding. By a reparametrization we obtain an isotopy $\overline{H}: M_g \times I \to M$ with $\overline{H}_0 = i$ and $\overline{H}_1(M_g) \subset U$. Since $M^n$ is closed, there is a diffeotopy $G: M \times I \to M$ such that

$$\begin{array}{c}
M_g\nearrow i \\
\downarrow \overline{H}_t \searrow G_t \\
M
\end{array}$$

commutes for all $t \in I$.

$g \circ G^{-1}$ is a regular homotopy from $g$ to $\tilde{g}$, and $U$ contains the double points $M_g$ of $\tilde{g}$. $\tilde{g}$ restricted to $M \setminus \tilde{U}$, $(\tilde{U} = \text{interior of } U)$ is an injective immersion of a compact manifold with boundary, hence an embedding. By [5] the regular homotopy class of $\tilde{g}|_{M \setminus \tilde{U}}$ is unique.

$\tilde{g} \circ K_{1 \partial D^n}: S^{n-1} \to \mathbb{R}^{2n-k}$ is an embedding with Smale invariant zero, thus regularly homotopic to the standard embedding $S^{n-1} \hookrightarrow \mathbb{R}^{2n-k}$. Define $\tilde{g}: S^n \to \mathbb{R}^{2n-k}$ by $\tilde{g} \circ k: D^n \to \mathbb{R}^{2n-k}$ on the northern hemisphere and a standard embedding on the southern hemisphere. Defining $I[g] := c[\tilde{g}]$, where $c$ means take the Smale invariant, we obtain our map $I: [M^n \hookrightarrow \mathbb{R}^{2n-k}] \to [S^n \hookrightarrow \mathbb{R}^{2n-k}]$.

Claim. (i) $I$ is well defined. (ii) $I$ is bijective.

(i) Let $f$, $g$ represent a class in $[M^n \hookrightarrow \mathbb{R}^{2n-k}]$. Proceeding as above we can regularly homotope $f$ and $g$ to an embedding outside a suitably chosen coordinate neighborhood of the basepoint of $M$. By [2] we may assume $f|_{M \setminus U} = g|_{M \setminus U}$ and a regular homotopy from $f$ to $g$ may be assumed to be an isotopy on $M \setminus U$. A regular homotopy from $f$ to $g$ is therefore given by a regular homotopy rel $\partial U$ from $f|_{U}$ to $g|_{U}$. Applying the above gluing construction, one obtains a regular homotopy from $f$ to $g$.

(ii) Let $j: M^n \hookrightarrow \mathbb{R}^{2n-k}$ be an embedding. An inverse $J$ to $I$ is given by $J(\varphi) = f_{\varphi} \# j$, where $f_{\varphi}: S^n \hookrightarrow \mathbb{R}^{2n-k}$ is an immersion with Smale invariant $\varphi$.

References


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