PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 4, April 1984

A NOTE ON QUASI-BUCHSBAUM RINGS

SHIRO GOTO[1]

ABSTRACT. In this paper the ubiquity of non-Buchsbaum but quasi-Buchsbaum rings is established. The result is stated as follows: Let $d \geqslant 3$ and $h_1, h_2, \ldots, h_{d-1} \geqslant 0$ be integers and assume that at least two of h_i's are positive. Then there exists a non-Buchsbaum but quasi-Buchsbaum local integral domain A of $\dim A = d$ and such that $l_A(H^i_{\mathfrak{m}}(A)) = h_i$ for all $1 \leqslant i \leqslant d-1$. Moreover if $h_1 = 0$ the ring A can be chosen to be normal.

1. Introduction. The purpose of this note is to give the following

THEOREM (1.1). *Let $d \geqslant 3$ and $h_1, h_2, \ldots, h_{d-1} \geqslant 0$ be integers and assume that at least two of h_i's are strictly positive. Then there exists a Noetherian integral local domain A such that*

(1) $\dim A = d$;

(2) $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(A) = (0)$ *and* $l_A(H^i_{\mathfrak{m}}(A)) = h_i$ *for all* $1 \leqslant i \leqslant d-1$;

(3) *A is not a Buchsbaum ring.*

Moreover if $h_1 = 0$, then the ring A can be chosen to be normal. (Here $l_A(H^i_{\mathfrak{m}}(A))$ denotes for each i the length of the ith local cohomology module of A relative to the maximal ideal $\mathfrak{m}$ in A.)

Now let us recall some definition. For a while let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated A-module of dimension d. Then we say that M is Buchsbaum (resp. quasi-Buchsbaum) if every system $a_1, a_2, \ldots, a_d$ of parameters for M (resp. at least one (and hence every) system $a_1, a_2, \ldots, a_d$ of parameters for M contained in $\mathfrak{m}^2$) forms a weak M-sequence, that is, the equality

$$(a_1, \ldots, a_{i-1})M : a_i = (a_1, \ldots, a_{i-1})M : \mathfrak{m}$$

holds for any $1 \leqslant i \leqslant d$, cf. [7] (resp. [9]). The ring A is called a Buchsbaum (resp. quasi-Buchsbaum) ring if A is a Buchsbaum (resp. quasi-Buchsbaum) module over itself.

Buchsbaum modules are of course quasi-Buchsbaum and a given finitely generated A-module M is quasi-Buchsbaum if and only if $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(M) = (0)$ for all $i \neq d$ [9]. Moreover, provided that $H^i_{\mathfrak{m}}(M) = (0)$ for $i \neq t, d$ where $t = \operatorname{depth}_A M$, a quasi-Buchsbaum A-module M is always Buchsbaum [8]. Nevertheless, without this

Received by the editors March 18, 1983.

1980 *Mathematics Subject Classification*. Primary 13H10, 13H15.

Key words and phrases. Buchsbaum rings and modules, quasi-Buchsbaum rings and modules, local cohomology.

[1]Partially supported by Grant-in-Aid for Co-operative Research.

0002-9939/84 $1.00 + $.25 per page

extra assumption, quasi-Buchsbaum modules are not necessarily Buchsbaum: the first counterexample was given by J. Stückrad in ring case of dimension 2 [**10**] and subsequently the author guaranteed, expanding Stückrad's idea, that for any integer $d \geqslant 2$, there exists a non-Buchsbaum but quasi-Buchsbaum local ring A of dimension d and with $H_{\mathfrak{m}}^i(A) = (0)$ for all $i \neq 0, 1, d$ [**2**]. However even in the latter examples, the rings A are still of depth 0 and almost all the local cohomology modules $H_{\mathfrak{m}}^i(A)$ vanish. On the contrast, according to Theorem (1.1), one can handle numerous non-Buchsbaum quasi-Buchsbaum normal rings with arbitrary local cohomology.

Quasi-Buchsbaum rings have just begun to be explored and several remarkable results are recently given by [**4**, **9** and **11**] on quasi-Buchsbaum rings and modules which are not necessarily Buchsbaum. As the theory of quasi-Buchsbaum rings itself is interesting enough to be developed, it is required, first of all, to establish the ubiquity of non-Buchsbaum but quasi-Buchsbaum rings and modules together with many examples. From this point of view our Theorem (1.1) may have some interest.

Our method of construction of examples is the same as in [**2**] which established the ubiquity of Buchsbaum rings. However for the present purpose we need a few preliminaries on quasi-Buchsbaum modules over polynomial rings that we shall summarize in the next section. The proof of Theorem (1.1) itself is simple and shall be given in §3.

2. Quasi-Buchsbaum modules over polynomial rings. In this section let $S = k[X_1, X_2, \ldots, X_n]$ be a polynomial ring with n ($n \geqslant 3$) variables over a field k and $\mathfrak{n} = S_+$, the irrelevant maximal ideal in S. We denote by $H_{\mathfrak{n}}^p(*)$, for each p, the pth local cohomology functor relative to $\mathfrak{n}$. Given a graded S-module M and an integer p, let $M(p)$ stand for the graded S-module which coincides with M as underlying modules and whose graduation is given by $[M(p)]_q = M_{p+q}$ for all $q \in \mathbf{Z}$.

A finitely generated graded S-module M is simply said to be Buchsbaum (resp. quasi-Buchsbaum) if the $S_{\mathfrak{n}}$-module $M_{\mathfrak{n}}$ is Buchsbaum (resp. quasi-Buchsbaum).

Let

$$\text{(F)} \qquad 0 \to F_n \to \cdots \to F_1 \xrightarrow{f_1} F_0 = S \to \underline{k} = S/\mathfrak{n} \to 0$$

be a graded minimal free resolution of the S-module $\underline{k} = S/\mathfrak{n}$. Recall that the resolution (F) can be identified with the Koszul complex of S generated by the elements $X_1, X_2, \ldots, X_n$. For each $0 \leqslant i \leqslant n-1$ we put

$$\begin{aligned} M_i &= \underline{k} \qquad (i = 0), \\ &= \mathfrak{n} \qquad (i = 1), \\ &= \operatorname{Ker}(F_{i-1} \to F_{i-2}) \qquad (n-1 \geqslant i \geqslant 2). \end{aligned}$$

Then we clearly have the following

LEMMA (2.1). *Let* $1 \leqslant i \leqslant n-1$. *Then*

(1) $\dim_S M_i = n$.

(2)
$$\begin{aligned} H_{\mathfrak{n}}^p(M_i) &= (0) \qquad (p \neq i, n), \\ &= \underline{k} \qquad (p = i). \end{aligned}$$

(3) $l_S(M_i/\mathfrak{n}M_i) = \binom{n}{i}$ *and* $\operatorname{rank}_S M_i = \binom{n-1}{i-1}$.

PROOF. Cf., e.g. [3, (3.1)].

We choose an S-free basis $\{e_i\}_{1 \leqslant i \leqslant n}$ of F_1 so that e_i's are homogeneous elements of degree 1 and $f_1(e_i) = X_i$ for any $1 \leqslant i \leqslant n$. Let $2 \leqslant t \leqslant n-1$ be an integer. For each subset J of $\{1,2,\ldots,n\}$ with $\#J = t$, we put

$$e_J = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_t}$$

in $F_t = \wedge^t F_1$ where $J = \{j_1, j_2, \ldots, j_t\}$ with $j_1 < j_2 < \cdots < j_t$.

We consider the canonical exact sequence

$$F_{t+1} \xrightarrow{f_{t+1}} F_t \xrightarrow{g_t} M_t \to 0$$

of graded S-modules and put

$$L_t = \mathfrak{n}M_t + \sum_J Sg_t(e_J),$$

where J runs over the subsets J of $\{1,2,\ldots,n\}$ with $\#J = t$ and such that $J \neq \{1,2,\ldots,t\}$.

LEMMA (2.2). (1) $\dim_S L_t = n$.

(2)
$$\begin{aligned} H^p_{\mathfrak{n}}(L_t) &= (0) \qquad (p \neq 1, t, n), \\ &= \underline{k} \qquad (p = t), \\ &= \underline{k}(-t) \qquad (p = 1). \end{aligned}$$

(3) L_t *is not a Buchsbaum S-module.*

PROOF. Since there is an exact sequence

$$0 \to L_t \to M_t \to \underline{k}(-t) \to 0$$

of graded S-modules, Assertions (1) and (2) follow from (2.1). Consider Assertion (4) and assume that L_t is a Buchsbaum S-module. Then we have the equality

$$l_S(L_t/\mathfrak{n}L_t) = I(L_t) + e_{L_t}(\mathfrak{n}),$$

where $e_{L_t}(\mathfrak{n})$ and $I(L_t)$ denote, respectively, the multiplicity of L_t relative to $\mathfrak{n}$ and the Buchsbaum invariant of L_t (cf. [7]). Hence

$$l_S(L_t/\mathfrak{n}L_t) = \left[\binom{n-1}{1} + \binom{n-1}{t}\right] + \binom{n-1}{t-1} = n - 1 + \binom{n}{t},$$

because $e_{L_t}(\mathfrak{n}) = \binom{n-1}{t-1}$ by (2.1)(3) and

$$I(L_t) = \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot l_S(H^i_{\mathfrak{n}}(L_t)) = \binom{n-1}{1} + \binom{n-1}{t}$$

by Assertion (2) (cf. [5, Satz 2]). Therefore L_t is minimally generated by $[\binom{n}{t} - 1] + n$ elements. On the other hand it is clear that the graded S-module L_t is generated by the $\binom{n}{t} - 1$ elements

$$\{g_t(e_J) \mid J \subset \{1,2,\ldots,n\} \text{ such that } \#J = t \text{ and } J \neq I\}$$

together with the following n elements

$$\{X_i g_t(e_I) \mid 1 \leqslant i \leqslant n\},$$

where $I = \{1, 2, \ldots, t\}$. Accordingly, these elements must form a minimal system of generators for L_t—this is, of course, not true since

$$X_{t+1} g_t(e_I) \in \sum_{i=1}^{t} S g_t(e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{t+1}).$$

Thus L_t is not a Buchsbaum S-module.

Let us recall the following

PROPOSITION (2.3) [**6** AND **8**]. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated A-module. If the canonical maps*

$$\operatorname{Ext}_A^i(A/\mathfrak{m}, M) \xrightarrow{h_M^i} H_{\mathfrak{m}}^i(M) = \varinjlim_t \operatorname{Ext}_A^i(A/\mathfrak{m}^t, M)$$

are surjective for all $i \neq \dim_A M$, then M is a Buchsbaum A-module. In case A is a regular local ring, the converse is also true.

Let $1 \leqslant s < t \leqslant n-1$ be integers. We put $L_{s,t} = L_t$ for $s = 1$. In case $s > 1$, we define $L_{s,t} = \operatorname{Ker}(G_{s-2} \to G_{s-3})$ where

$$G_{s-2} \to \cdots \to G_1 \to G_0 \to L_{t-s+1} \to 0$$

is a part of a graded minimal free resolution of L_{t-s+1}. Notice that if $s \geqslant 2$, there exists an exact sequence

$$(\#) \qquad 0 \to L_{s,t} \to G_{s-2} \to L_{s-1,t-1} \to 0$$

of graded S-modules.

LEMMA (2.4). (1) $\dim_S L_{s,t} = n$.

(2)
$$\begin{aligned} H_{\mathfrak{n}}^p(L_{s,t}) &= (0) \qquad (p \neq s, t, n), \\ &= \underline{k}(s-t-1) \qquad (p = s), \\ &= \underline{k} \qquad (p = t). \end{aligned}$$

(3) *$L_{s,t}$ is not a Buchsbaum S-module.*

PROOF. Assertion (1) is obvious and Assertion (2) follows by induction on s from the above exact sequence $(\#)$. Consider Assertion (3). By (2.2)(3) we may assume that $s \geqslant 2$ and that our assertion is true for $s-1$. Apply functors $\operatorname{Ext}_S^i(S/\mathfrak{n}, *)$ and $H_{\mathfrak{n}}^i(*)$ to the exact sequence $(\#)$ and we get, for each $0 \leqslant i \leqslant n-1$, a commutative square

$$(\#\#) \qquad \begin{array}{ccc} \operatorname{Ext}_S^{i-1}(S/\mathfrak{n}, L_{s-1,t-1}) & \cong & \operatorname{Ext}_S^i(S/\mathfrak{n}, L_{s,t}) \\ \downarrow h_{L_{s-1,t-1}}^{i-1} & & \downarrow h_{L_{s,t}}^i \\ H_{\mathfrak{n}}^{i-1}(L_{s-1,t-1}) & \cong & H_{\mathfrak{n}}^i(L_{s,t}) \end{array}$$

of graded S-modules, where $h_{L_{s-1,t-1}}^{i-1}$ and $h_{L_{s,t}}^i$ stand for canonical homomorphisms. As

$$H_{\mathfrak{n}}^{n-1}(L_{s-1,t-1}) = (0)$$

by Assertion (2), we see by (2.3) and by induction hypothesis that the map $h_{L_{s-1,t-1}}^{i-1}$ can not be surjective for some $i \leqslant n-1$; hence the commutative diagram $(\#\#)$

guarantees that $h^i_{L_{s,t}}$ is not surjective. Thus again by (2.3), we conclude that $L_{s,t}$ is not a Buchsbaum $\underline{S}$-module.

Now let $1 \leqslant s < t \leqslant n-1$ be integers. Let $h_0, h_1, \ldots, h_{n-1} \geqslant 0$ be integers such that h_s and h_t are strictly positive. We put

$$\begin{aligned} h &= \min\{h_s, h_t\}, \\ u &= s \qquad (h_s \geqslant h_t), \\ &= t \qquad (h_s < h_t) \end{aligned}$$

and

$$E = \sum_{\substack{0 \leqslant i \leqslant n-1 \\ \text{such that} \\ i \neq s,t}} M_i^{h_i} \oplus M_u^{h_u - h} \oplus L_{s,t}^h,$$

where M^r denotes, for a given S-module M and an integer $r \geqslant 0$, the direct sum of r copies of M. Then by (2.1) and (2.4) we immediately get the following

THEOREM (2.5). (1) $\dim_S E = n$.

(2) *Let $0 \leqslant i < n$ be an integer. Then*

$$\begin{aligned} H^i_{\mathfrak{n}}(E) &= \underline{k}^{h_i} \qquad (i \neq s), \\ &= \underline{k}^{h_s - h} \oplus [\underline{k}(s-t-1)]^h \qquad (i = s). \end{aligned}$$

(3) *E is a quasi-Buchsbaum S-module but not Buchsbaum.*

3. Proof of Theorem (1.1). Let $d \geqslant 3$ and $h_1, h_2, \ldots, h_{d-1} \geqslant 0$ be integers. Assume that h_s and h_t are strictly positive for some s and t ($1 \leqslant s < t \leqslant d-1$). We put $n = d + 2$ and

$$\begin{aligned} h'_i &= 0 \qquad (i = 0, 1, d+1), \\ &= h_{i-1} \qquad (d \geqslant i \geqslant 2). \end{aligned}$$

Let $S = k[X_1, X_2, \ldots, X_n]$ be a polynomial ring with n variables over an infinite field k and consider the graded S-module E obtained by (2.5) for the above integers h'_i ($0 \leqslant i \leqslant n-1$). Then as $E_{\mathfrak{p}}$ is a free $S_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ of S such that $\mathfrak{p} \neq \mathfrak{n}$ and as $\operatorname{depth}_{S_{\mathfrak{n}}} E_{\mathfrak{n}} \geqslant 2$, we get by virtue of [**1**, Theorem B] a short exact sequence

$$0 \to F \to E \to P(r) \to 0$$

of graded S-modules, where F is a graded free S-submodule of E, P is a graded prime ideal of S with $ht_S P = 2$ and r is an integer. (Notice that if $h_1 = 0$, the prime ideal P can be taken so that the ring S/P is normal.)

We put $R = S/P$ and $A = S_{\mathfrak{n}}/PS_{\mathfrak{n}}$. Then by the following two exact sequences

$$0 \to P \to S \to R \to 0 \quad \text{and} \quad 0 \to F \to E \to P(r) \to 0$$

of graded S-modules, we get isomorphisms

$$H^i_{\mathfrak{n}}(R) \cong H^{i+1}_{\mathfrak{n}}(P) \quad \text{and} \quad H^{i+1}_{\mathfrak{n}}(E) \cong [H^{i+1}_{\mathfrak{n}}(P)](r)$$

$(1 \leqslant i \leqslant d-1)$ of local cohomology; hence for each $1 \leqslant i \leqslant d-1$,

$$\begin{aligned} H^i_{\mathfrak{n}}(R) &= [\underline{k}(-r)]^{h_i} \qquad (i \neq s), \\ &= [\underline{k}(-r)]^{h_s-h} \oplus [\underline{k}(s-t-r-1)]^{h} \qquad (i = s), \end{aligned}$$

where $h = \min\{h_s, h_t\}$ (cf. (2.5)). Moreover, by the same exact sequences, we get for each $i \leqslant d-1$ commutative squares

$$\begin{array}{ccc} \mathrm{Ext}^i_S(S/\mathfrak{n}, R) & \cong & \mathrm{Ext}^{i+1}_S(S/\mathfrak{n}, P) \\ \downarrow h^i_R & & \downarrow h^{i+1}_P \\ H^i_{\mathfrak{n}}(R) & \cong & H^{i+1}_{\mathfrak{n}}(P) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Ext}^{i+1}_S(S/\mathfrak{n}, E) & \cong & [\mathrm{Ext}^{i+1}_S(S/\mathfrak{n}, P)](r) \\ \downarrow h^{i+1}_E & & \downarrow h^{i+1}_P(r) \\ H^{i+1}_{\mathfrak{n}}(E) & \cong & [H^{i+1}_{\mathfrak{n}}(P)](r) \end{array}$$

of graded S-modules. As $H^{d+1}_{\mathfrak{n}}(E) = (0)$ by our choice and as E is not a Buchsbaum S-module by (2.5)(3), we see by (2.3) that the canonical map h^{i+1}_E is not surjective for some $i \leqslant d-1$. Therefore, by the above commutative squares we immediately find that for some $0 \leqslant i \leqslant d-1$, the map h^i_R is not surjective. As $\dim R = d$, this guarantees again by (2.3) that R is not a Buchsbaum ring. Thus the local ring $A = R_{\mathfrak{n}}$ satisfies all the requirements in Theorem (1.1), which completes the proof of Theorem (1.1).

References

1. E. G. Evans, Jr. and P. A. Griffith, *Local cohomology modules for normal domains*, J. London Math. Soc. **19** (1979), 277–284.

2. S. Goto, *On Buchsbaum local rings*, R.I.M.S. Kôkyuroku **374** (1980), 58–74. (Japanese)

3. ______, *On Buchsbaum rings*, J. Algebra **67** (1980), 272–279.

4. S. Goto and N. Suzuki, *Index of reducibility of parameter ideals in a local ring*, J. Algebra (to appear).

5. B. Renschuch, J. Stückrad and W. Vogel, *Weitere Bemerkungen zu einem Problem der Schnittheorie und ein Mass von A. Seidenberg für die Imperfektheit*, J. Algebra **37** (1975), 447–471.

6. J. Stückrad, *Über die kohomologische Charakterisierung von Buchsbaum-Moduln*, Math. Nachr. **95** (1980), 265–272.

7. J. Stückrad and W. Vogel, *Eine Verallgemeinerung der Cohen-Macaulay Ringe und Anwendungen auf ein Problem der Multiplizitätstheorie*, J. Math. Kyoto Univ. **13** (1973), 513–528.

8. ______, *Toward a theory of Buchsbaum singularities*, Amer. J. Math. **100** (1978), 727–746.

9. N. Suzuki, *On a basic theorem for quasi-Buchsbaum modules*, Bull. Dept. General Ed. Shizuoka Coll. Pharmacy **11** (1982), 33–40.

10. W. Vogel, *A non-zero-divisor characterization of Buchsbaum modules*, Michigan Math. J. **28** (1981), 147–152.

11. K. Yamagishi, *On unconditioned strong d-sequences* (to appear).

Department of Mathematics, Nihon University, Setagaya - ku, Tokyo 156, Japan