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CROSS PRODUCTS OF STRONGLY MORITA EQUIVALENT C^*-ALGEBRAS[1]

RAUL E. CURTO, PAUL S. MUHLY AND DANA P. WILLIAMS

ABSTRACT. Suppose that a locally compact group G acts on strongly Morita equivalent C^*-algebras A and B and let $A \rtimes G$ and $B \rtimes G$ denote the corresponding crossed products. We present conditions which imply that $A \rtimes G$ and $B \rtimes G$ are also strongly Morita equivalent and we apply our result to improve upon known theorems concerning strong Morita equivalence between certain transformation group C^*-algebras.

1. Denote the action of G on A by α and that of G on B on β. Suppose, too, that X is a complete A-B-equivalence bimodule in the sense of [**4**, p. 287].

THEOREM 1. *If there is a strongly continuous action of G on X*, $\{\tau_t\}_{t\in G}$, *such that for* $a \in A$, $b \in B$ *and* $x, y \in X$, *the following equations*, (i) *and* (ii) *are satisfied, then* $A \rtimes G$ *and* $B \rtimes G$ *are strongly Morita equivalent.*

(i) $\langle \tau_t x, \tau_t y\rangle_A = \alpha_t(\langle x, y\rangle_A)$ *and*

(ii) $\langle \tau_t x, \tau_t y\rangle_B = \beta_t(\langle x, y\rangle_B)$.

PROOF. Since $\langle x, y\rangle_A \langle u, v\rangle_A = \langle x\langle y, u\rangle_B, v\rangle$ for x, y, u, and $v \in X$, it follows from the fact that each α_t is an automorphism that

$$\tau_t(x\langle y, u\rangle_B) = \tau_t(x)\beta_t(\langle y, u\rangle_B).$$

Since range of $\langle \cdot, \cdot\rangle_B$ spans a dense subset of B, we notice that $\tau_t(x \cdot b) = \tau_t(x)\beta_t(b)$ for each $x \in X$ and $b \in B$. Similarly, $\tau_t(a \cdot x) = \alpha_t(a)\tau_t(x)$.

Let C be the linking algebra for A and B constructed in [**1**]. Then C consists of matrices $\left(\begin{smallmatrix} a & x \\ \tilde{y} & b\end{smallmatrix}\right)$ where $a \in A$, $b \in B$, $x, y \in X$, and where $\tilde{y}$ denotes the image of y in the dual B-A-equivalence bimodule. Since X is assumed complete, C is a C^*-algebra and may be faithfully represented by bounded operators on the (right) B-rigged space $M = X + B$ (with B-valued inner product $\langle \left(\begin{smallmatrix} x \\ b\end{smallmatrix}\right), \left(\begin{smallmatrix} y \\ c\end{smallmatrix}\right)\rangle_B = \langle x, y\rangle_B + b^*c$) according to the formula

$$\begin{pmatrix} a & x \\ \tilde{y} & b\end{pmatrix}\begin{pmatrix} z \\ c\end{pmatrix} = \begin{pmatrix} az + xc \\ \langle y, z\rangle_B + bc\end{pmatrix}.$$

Define $\{\gamma_t\}_{t\in G}$ on C by the formula

$$\gamma\begin{pmatrix} a & x \\ \tilde{y} & b\end{pmatrix} = \begin{pmatrix} \alpha_t(a) & \tau_t(x) \\ (\tau_t y)^{\sim} & \beta_t(b)\end{pmatrix}$$

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and use hypotheses (i) and (ii), as well as the covariance property demonstrated above, to check that $t \to \gamma_t$ is a homomorphism of G into the *-automorphisms of C. To see that $\{\gamma_t\}_{t\in G}$ is strongly continuous, it suffices to check that for $c \in C$, $\lim_{t\to e}\gamma_t(c)\binom{x}{b} = c\binom{x}{b}$ uniformly in the B-norm on M as $\binom{x}{b}$ ranges over bounded subsets of M. This is most easily done for elements in C with one nonzero entry and we omit the details. As in [**1**], let $p = \begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_B \end{pmatrix}$ where id_A and id_B are the identity maps of A and B onto themselves. Then p and q are complementary full projections in the multiplier algebra of C, $M(C)$, and the maps i_A: $A \to C$ and i_B: $B \to C$ defined by $i_A(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $i_B(b) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ are isometric imbeddings of A and B into C with images pCp and qCq respectively. Moreover, p and q are invariant under $\{\gamma_t\}_{t\in G}$, and the maps i_A and i_B are equivariant, i.e., $i_A \circ \alpha_t = \gamma_t \circ i_A$ and $i_B \circ \beta_t = \gamma_t \circ i_B$. If we form $C \rtimes G$ using $\{\gamma_t\}_{t\in G}$, and if we view p and q as elements in $M(C \rtimes G)$, then it is easy to see that p and q are complementary full projections in $M(C \rtimes G)$. Moreover, by equivariance, the maps i_A and i_B extend in the obvious way to imbeddings of $A \rtimes G$ and $B \rtimes G$ into $C \rtimes G$ with images $p(C \rtimes G)p$ and $q(C \rtimes G)q$, respectively. We conclude from [**1**, Theorem 1.1] that $A \rtimes G$ and $B \rtimes G$ are strongly Morita equivalent.

2. In [**5**], Rieffel describes a number of circumstances under which certain transformation group C^*-algebras are seen to be strongly Morita equivalent. His Situation 10 is the most general in the sense that all of the others are special cases of it. We use Theorem 1 to show how to improve upon Situation 10 and how to derive the improvement from Rieffel's Situation 2. Following [**5**], suppose that H and K are locally compact groups acting on a locally compact space P. We assume that H and K act freely, that for each action compact sets are wandering, and that the actions commute. The wandering hypothesis implies that the orbit spaces P/H and P/K are locally compact, and since the actions commute, there is a natural action of K on P/H and one of H on P/K. Consequently, we may form the transformation group C^*-algebras, $C^*(H, P/K)$ and $C^*(K, P/H)$. On the other hand, since the actions commute, we may view $H \times K$ as acting on P in the obvious way, and so we may form $C^*(H \times K, P)$.

THEOREM 2. *The three algebras*, $C^*(H, P/K)$, $C^*(K, P/H)$ *and* $C^*(H \rtimes K, P)$, *are strongly Morita equivalent.*

PROOF. Situation 2 of [**5**] asserts that since the action of K on P satisfies the wandering hypothesis, $C^*(K, P)$ and $C_0(P/K)$ are strongly Morita equivalent and that a natural $C^*(K, P) - C_0(P/K)$-equivalence bimodule X is the completion of $C_c(P)$ in the $C_0(P/K)$-valued inner product. The C^*-algebra $C^*(K, P)$ is a completion of $C_c(K \times P)$ and one obtains an action of H on $C^*(K, P)$ by defining it first on $C_c(K \times P)$ by the formula $(\alpha_h\Phi)(k, p) = \Phi(k, h^{-1}p)$, $\Phi \in C_0(K \times P)$. Also, H acts on $C_0(P/K)$ according to the formula $(\beta_h\Omega)(\dot{p}) = \Omega((h^{-1}p)^{\dot{}})$, $\Omega \in C_0(P/K)$, where $\dot{p}$ denotes the image of p in P under the quotient map onto P/K. Finally, for $h \in H$ and $f \in C_c(p)$, we define $(\tau_h f)(p) = f(h^{-1}p)$. Then it is routine to check the formulas in Theorem 1 at the level of functions in the appropriate spaces, $C_c(K \times P)$,

$C_c(P)$, and $C_0(P/K)$. But it is also routine to check that $\{\tau_h\}_{h\in H}$ is strongly continuous in the norm on $C_c(P)$ determined by the $C_0(P/K)$-valued inner product. Thus $\{\tau_t\}_{h\in H}$ extends to all of X and the hypotheses of Theorem 1 are satisfied. By Theorem 1, $C^*(K, P) \rtimes H$ and $C_0(P/K) \rtimes H$ are strongly Morita equivalent, but since these algebras are isomorphic to $C^*(H \times K, P)$ and $C^*(H, P/K)$, respectively, we conclude that $C^*(H \times K, P)$ and $C^*(H, P/K)$ are strongly equivalent. Since the equivalence of $C^*(H \times K, P)$ and $C^*(K, P/H)$ may be proved similarly, the proof is complete.

REMARK. It may appear that we have given an entirely elementary proof of Rieffel's Situation 10 avoiding his slick but unintuitive argument. However, this is not quite the case; one still has to verify Situation 2. To be sure, Rieffel's argument simplifies somewhat in the context of Situation 2, but not materially. Another argument for Situation 2 may be fashioned easily from Green's Theorem 14 in [**2**] and Raeburn's Proposition 1.1 in [**3**]. While this approach is perhaps more intuitive than Rieffel's, it certainly is much longer.

ADDED IN PROOF. We recently received the preprint, *Crossed products and Morita equivalence*, by F. Combes, in which he also proves Theorem 1.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242 (Current address of Raul E. Curto and Paul S. Muhly)

DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843 (Current address of Dana P. Williams)