A NOTE ON DISJOINTNESS PRESERVING OPERATORS

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Abstract. In this paper we present some results concerning the automatic order boundedness of disjointness preserving operators on Riesz spaces (vector lattices).

Let \( L \) and \( M \) be Archimedean Riesz spaces. The linear mapping \( T \) from \( L \) into \( M \) is called disjointness preserving if \( Tf \perp Tg \) whenever \( f \perp g \) in \( L \). Observe that a positive linear mapping \( T \) is disjointness preserving iff \( T \) is a Riesz homomorphism.

The purpose of this note is to prove a basic result (Theorem 2) concerning disjointness preserving operators, which has many applications to the automatic order boundedness problem for disjointness preserving operators. First of all it provides a short and simple proof of a recent result of Yu. A. Abramovich [1] to the effect that a disjointness preserving operator \( T \) with the additional property that \( \inf_n(\| T_{f_n} \| + \| T_{g_n} \|) = 0 \) in \( M \) whenever \( f_n, g_n \to 0 \) (r.u.) in \( L \), is order bounded. We note that a similar result for band preserving operators on Archimedean Riesz spaces was proved by S. J. Bernau [6] (recall that the linear operator \( T \) from \( L \) into itself is called band preserving if \( f \perp g \) implies that \( Tf \perp g \)).

Furthermore, Theorem 2 has as an immediate corollary that any order bounded disjointness preserving operator \( T \) can be written as the difference of two Riesz homomorphisms, a result due to M. Meyer [10].

Finally we shall use Theorem 2 to show that any disjointness preserving operator from \( L \) into \( M \) is order bounded on some order dense ideal in \( L \), whenever \( L \) is uniformly complete and \( M \) satisfies some additional conditions (e.g. if \( M \) is a normed Riesz space). In particular, it follows from a combination of Proposition 6 and Theorem 8 that any band preserving operator on a Banach lattice is order bounded (and hence norm bounded), which is a result of Yu. Abramovich, A. I. Veksler and A. V. Koldunov [2].

For terminology used and properties of Riesz spaces not explained or proved in this paper, we refer to [4 or 9]. We start with a lemma.

Lemma 1. Let \( L \) be an Archimedean Riesz space and let \( n \) be a fixed natural number. If \( 0 \leq u \leq e \) in \( L \), then there exist \( 0 \leq p_k \in L \) \((k = 0, 1, \ldots, 2n)\) and \( 0 \leq v, w \in L \) such that

(i) \( \sum_{k=0}^{2n} p_k = u \),
(ii) \( |v - u| \leq \epsilon/n \) and \( |w - u| \leq \epsilon/n \),

(iii) \( v - ke/n \perp p_{2k} \) for \( k = 0, 1, \ldots, n \) and \( w - (2k - 1)e/2n \perp p_{2k-1} \) for \( k = 1, \ldots, n \).

**Proof.** For \( k = 1, 2, \ldots, 2n \) put \( y_k = [4nu - (2k - 1)e]^+ \wedge e \). Define

\[
v = \frac{1}{n} \sum_{k=1}^{n} y_{2k}, \quad w = \frac{1}{2n} y_1 + \frac{1}{n} \sum_{k=2}^{n} y_{2k-1},
\]

and \( p_k = [4nu - 2ke]^+ \wedge u - [4nu - 2(k + 1)e]^+ \wedge u \) \((k = 0, 1, \ldots, 2n)\). We shall show that \( \{ p_k : k = 0, 1, \ldots, 2n \} \), \( v \) and \( w \) fulfill the requirements. First of all, since \( p_0 = u - (4nu - 2e)^+ \wedge u \) and \( p_{2n} = 0 \), it is clear that (i) holds.

In order to prove (ii), observe that

\[
|v - u| \leq \sum_{k=1}^{n} \frac{1}{n} y_{2k} - \left[ u - \left( \frac{k - 1}{n} \right) e \right]^+ \wedge \left( \frac{1}{n} e \right),
\]

and so

\[
|v - u| \leq \sum_{k=1}^{n} \frac{1}{n} y_{2k} - \left[ u - \left( \frac{k - 1}{n} \right) e \right]^+ \wedge \left( \frac{1}{n} e \right).
\]  

(1)

Since each term on the right-hand side is majorized by \( \epsilon/n \), it is sufficient to show that these terms are mutually disjoint. Take \( 1 \leq k < l \leq n \). Since

\[
y_{2l} \leq 4n[u - ((l - 1)/n)e]^+,
\]

it is sufficient to show that

\[
\frac{1}{n} y_{2k} - \left[ u - \left( \frac{k - 1}{n} \right) e \right]^+ \wedge \left( \frac{1}{n} e \right),
\]

and hence we have to prove that

\[
[4nu - (4k - 1)e]^+ \wedge e \leq [nu - (k - 1)e]^+ \wedge e \perp (nu - ke)^+.
\]

Note that for \( f, g \in L \) we have \( |f \wedge e - g \wedge e| \leq (e - f)^+ \vee (e - g)^+ \), \((e - f)^+ = (e - f)^+ \wedge e \). Now

\[
(e - [4nu - (4k - 1)e]^+)^+ = (4ke - 4nu)^+ \wedge e \perp (nu - ke)^+,
\]

and similarly

\[
(e - [nu - (k - 1)e]^+)^+ = (ke - nu)^+ \wedge e \perp (nu - ke)^+.
\]

Hence the terms in the sum (1) are mutually disjoint and therefore \( |v - u| \leq \epsilon/n \). The proof that \( |w - u| \leq \epsilon/n \) follows the same lines.

In order to prove (iii) we first show that \( e - y_j \perp p_k \) for \( l \leq k \) and that \( y_j \perp p_k \) for \( l \geq k + 2 \). Suppose that \( l \leq k \), then

\[
e - y_j = e - [4nu - (2l - 1)e]^+ \wedge e = \left( e - [4nu - (2l - 1)e]^+ \right)^+ = (2le - 4nu)^+ \wedge e \leq (2ke - 4nu)^+ \perp (4nu - 2ke)^+ \wedge u,
\]
and hence $e - y_i \perp p_k$. Now suppose that $l \geq k + 2$, then
\[ p_k \leq u - [4nu - 2(k + 1)]^+ \land u = (u - [4nu - 2(k + 1)e]^+) \]
\[ = [2(k + 1)e - (4n - 1)u]^+ \land u \leq [(2l - 2)e - (4n - 1)u]^+ \]
\[ \leq [(2l - 1)e - 4nu]^+ \perp [4nu - (2l - 1)e]^+ , \]
and hence $p_k \perp y_l$. Now
\[ v - \frac{k}{n}e = \frac{1}{n} \sum_{i=1}^{k} (y_{2i} - e) + \frac{1}{n} \sum_{i=k+1}^{n} y_{2i}, \]
and by the above $y_{2i} - e \perp p_{2k} (1 \leq i \leq k)$ and $y_{2i} \perp p_{2k} (k + 1 \leq i \leq n)$. Hence $v - ke/n \perp p_{2k}$ for $k = 0, 1, \ldots, n$. In like manner it is shown that $w - (2k - 1)e/2n \perp p_{2k-1} (k = 1, 2, \ldots, n)$.

**Theorem 2.** Let $L$ and $M$ be Archimedean Riesz spaces and $T$ a disjointness preserving operator from $L$ into $M$. If $0 \leq u < e$ in $L$, then there exist two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$, contained in the ideal generated by $e$ in $L$, such that $f_n, g_n \to 0$ (e-uniformly) and $|Tu - Te|^+ \leq |Tf_n|^+ + |Tg_n|$ for all $n$.

**Proof.** Take $0 \leq u < e$ in $L$. By the above lemma there exist for each $n = 1, 2, \ldots$ elements $0 \leq p_{nk} \in L (k = 0, 1, \ldots, 2n)$ and $0 \leq v_n, w_n \in L$ such that $v_n$ and $w_n$ are contained in the ideal generated by $e$, and
1. $\sum_{k=0}^{2n} p_{nk} = u$,
2. $|v_n - u| \leq e/n, |w_n - u| \leq e/n$,
3. $v_n - ke/n \perp p_{n,2k}$ ($k = 0, 1, \ldots, n$) and $w_n - (2k - 1)e/2n \perp p_{n,2k-1}$ ($k = 1, 2, \ldots, n$).

Since $T$ is disjointness preserving, it follows from (iii) that
\[ Tvn - \frac{k}{n}Te \perp Tp_{n,2k} \quad \text{and} \quad Tw_n - \frac{2k - 1}{2n}Te \perp Tp_{n,2k-1} \quad (0 \leq k \leq n). \]
Furthermore, $|Tv_n - |Te||^+ \leq |Tv_n - kTe/n|$ and $|Tw_n - |Te||^+ \leq |Tw_n - (2k - 1)Te/2n|$, which implies that $|Tv_n - |Te||^+ \land (|Tw_n - |Te||^+ \perp Tp_{nk}$, i.e., that
\[ (|Tv_n| \land |Tw_n - |Te||^+ \land |Tp_{nk}| = 0 \quad \text{for all} \quad k = 0, 1, \ldots, 2n. \]
Now it follows from (i) that $|Tv_n| \land |Tw_n - |Te||^+ \land |Tu| = 0$.

For all $n = 1, 2, \ldots$ we now have
\[ (|Tu - |Te||^+ \leq |T(u - v_n)| + (|Tv_n - |Te||^+ \]
and
\[ (|Tu - |Te||^+ \leq |T(u - w_n)| + (|Tw_n - |Te||^+ , \]

hence
\[ (|Tu - |Te||^+ \leq \{|T(u - v_n)| + (|Tv_n - |Te||^+ \}
\[ \land \{ |T(u - w_n)| + (|Tw_n - |Te||^+ \}
\[ \leq |T(u - v_n)| + |T(u - w_n)| + (|Tv_n| \land |Tw_n - |Te||^+ . \]
Since $|Tu| - |Te| \leq |Tu|$, this implies that $|Tu| - |Te| \leq |T(u - v_n)| + |T(u - w_n)|$, and so we can take $f_n = u - v_n$ and $g_n = u - w_n$ for $n = 1, 2, \ldots$.

The next corollary includes Abramovich's results [1, Proposition B].

**Corollary 3.** Let $L$ and $M$ be Archimedean Riesz spaces and $T$ a disjointness preserving operator from $L$ into $M$. If $T$ has the property that $\inf_n(|Tf_n| + |Tg_n|) = 0$ whenever $f_n, g_n \to 0$ (r.u.) in $L$, then $0 \leq u \leq v$ in $L$ implies that $|Tu| \leq |Te|$ (and hence $T$ is order bounded).

**Corollary 4** (see [10 and 11]). Let $L$ and $M$ be Archimedean Riesz spaces and let $T$ be an order bounded disjointness preserving operator from $L$ into $M$. Then $T = T^+ - T^-$, where $T^+$ and $T^-$ are Riesz homomorphisms and $T^+ u = (Tu)^+$, $T^- u = (Tu)^-$ for all $0 \leq u \in L$.

**Proof.** Let $M^*$ be the Dedekind completion of $M$ and consider $T$ as an operator from $L$ into $M^*$. Since $T$ is order bounded, the absolute value $|T|$ exists as mapping from $L$ into $M^*$. For each $0 \leq u \in L$ we have $|T|(u) = \sup \{|Tf| : |f| \leq u\}$. Since $T$ is order bounded it is clear that the conditions of Corollary 3 are fulfilled, and so $|Tf| = |Tf| \leq |Tu|$ for all $f \in L$ with $|f| \leq u$. Hence $|T|(u) = |Tu|$, which shows that $|T|$ maps $L$ into $M$. Moreover, it follows from $T^+ = \frac{1}{2}(|T| + T)$ and $T^- = \frac{1}{2}(|T| - T)$ that $T^+ u = (Tu)^+$ and $T^- u = (Tu)^-$ for all $0 \leq u \in L$. Finally it is clear that $|T|$, $T^+$ and $T^-$ are Riesz homomorphisms.

**Remark 5.** Let $L$ be a Riesz space with the principal projection property and let $T$ be a disjointness preserving operator from $L$ into an Archimedean Riesz space $M$. If $0 \leq u \leq e$ in $L$ then there exists a sequence $\{f_n: n = 1, 2, \ldots\}$ in $L$ such that $f_n \to 0$ (r.u.) and $|Tu| \leq |Te|$ for all $n$. Indeed, by the Freudenthal spectral theorem [9, Theorem 40.2] there exists a sequence $\{s_n: n = 1, 2, \ldots\}$ in $L$ such that $0 \leq s_n \leq u$ (e-uniformly), where each $s_n$ is a sum of the form $\Sigma_{i=1}^k \alpha_i p_i$ with $0 \leq \alpha_i \leq 1 (i = 1, \ldots, k)$ and $p_1, \ldots, p_k$ mutually disjoint components of $e$. Using that $T$ is disjointness preserving, it follows easily that $|Ts_n| \leq |Te|$ and so $|Tu| \leq |Tu - s_n|$ (n = 1, 2, \ldots). Hence we can take $f_n = u - s_n$. It follows, in particular, that if in this case $T$ has the additional property that $\inf_n(|Tf_n| = \inf_n(|Te|)$ whenever $f_n \to 0$ (r.u.) in $L$, then $T$ is order bounded (and $0 \leq u \leq e$ in $L$ implies that $|Tu| \leq |Te|$).

As observed by Abramovich [1, §2], if $L$ and $M$ are normed Riesz spaces and $T$ is a norm bounded disjointness preserving operator from $L$ into $M$, then $f_n, g_n \to 0$ (r.u.) in $L$ implies that $\inf_n(|Tf_n| + |Tg_n|) = 0$ in $M$, and so $T$ is order bounded. This was also proved in the case when $L$ is a Dedekind $\sigma$-complete Banach lattice by W. Arendt [5, Theorem 2.5].

A linear operator $T$ from an Archimedean Riesz space $L$ into itself is called **band preserving** if $T(B) \subseteq B$ for all bands $B$ in $L$ (equivalently, $Tf \perp g$ whenever $f \perp g$ in $L$). Since a band preserving operator is clearly disjointness preserving, all of the above results apply to band preserving operators. An order bounded band preserving operator is called an **orthomorphism**. Band preserving operators and orthomorphisms have been studied extensively (see e.g. [2, 6, 7, 8, 12, 13 and 14]).
We shall present some other applications of Theorem 2. In particular we will prove that under certain conditions on the spaces \( L \) and \( M \), any disjointness preserving operator from \( L \) into \( M \) is order bounded on some order dense ideal in \( L \).

Let \( L \) and \( M \) be Archimedean Riesz spaces and let \( T \) be a linear operator from \( L \) into \( M \). If \( A \) and \( B \) are ideals in \( L \) such that the restrictions \( T|A \) and \( T|B \) are order bounded, then \( T|(A + B) \) is order bounded as well. Hence, if \( A_T \) is the union of all ideals \( A \) in \( L \) with the property that \( T|A \) is order bounded, then \( A_T \) is an ideal in \( L \) and \( T|A_T \) is order bounded. Clearly, \( A_T \) is the largest ideal in \( L \) on which \( T \) is order bounded. If \( T \) is disjointness preserving, then it follows immediately from Corollary 3 that \( |Tu| \leq |Tv| \) whenever \( 0 \leq u \leq v \in A_T \). For band preserving operators we have the following interesting result.

**Proposition 6.** If \( T \) is a band preserving operator on the Archimedean Riesz space \( L \), then \( A_T \) is a band.

**Proof.** Take \( 0 \leq u \leq e \in A_T^{dd} \) and suppose that \( (|Tu| - |Te|)^+ > 0 \). Then there exists \( v \in A_T \) such that \( 0 < v \leq (|Tu| - |Te|)^+ \). Let \( n \) be a natural number such that \( (nv - e)^+ > 0 \) and put \( p = nv \wedge u \) and \( q = nv \wedge e \). Then \( 0 \leq p \leq q \in A_T \) and so \( |Tp| \leq |Tq| \). Since \( T \) is band preserving, it follows from \( u - p \perp (nv - e)^+ \) and \( e - q \perp (nv - e)^+ \) that \( Tu -Tp \perp (nv - e)^+ \) and \( Te -Tq \perp (nv - e)^+ \). Now it is easy to see that

\[
(|Tu| - |Te|)^+ - (|Tp| - |Tq|)^+ \perp (nv - e)^+
\]

and hence \( (|Tu| - |Te|)^+ \perp (nv - e)^+ \). By the choice of \( v \) this implies that \( (nv - e)^+ = 0 \), which is a contradiction. We thus have proved that \( |Tu| \leq |Te| \) whenever \( 0 \leq u \leq e \in A_T^{dd} \). Therefore \( T \) is order bounded on \( A_T^{dd} \). Now it follows from the definition of \( A_T \) that \( A_T = A_T^{dd} \); hence \( A_T \) is a band.

In order to prove the next result we need a lemma.

**Lemma 7.** Let \( L \) and \( M \) be Archimedean Riesz spaces and assume that \( L \) is uniformly complete. Let \( T \) be a disjointness preserving operator from \( L \) into \( M \). Suppose \( 0 \leq u_n \leq e_n \leq e \ (n = 1,2,\ldots) \) in \( L \) such that \( \{e_n\}_{n=1}^\infty \) is a disjoint sequence and put \( w_n = (|Tu_n| - |Te_n|)^+ \) for all \( n \). Then \( \{w_n\}_{n=1}^\infty \) is a disjoint sequence in \( M \) with the property that \( \{\lambda_n w_n\}_{n=1}^\infty \) is order bounded in \( M \) for any sequence \( \{\lambda_n\}_{n=1}^\infty \) of positive real numbers.

**Proof.** By Theorem 2 there exist for each \( n \) two sequences \( \{f_{nk}\}_{k=1}^\infty \) and \( \{g_{nk}\}_{k=1}^\infty \) in the ideal generated by \( e_n \) such that \( f_{nk}, g_{nk} \to 0 \) (\( e \)-uniformly) as \( k \to \infty \) and \( 0 \leq w_n \leq (|Tf_{nk}| + |Tg_{nk}|) \) \((k = 1,2,\ldots) \) for all \( n \). Take positive real numbers \( \{\lambda_n\}_{n=1}^\infty \). For each \( n \) there exists a natural number \( k_n \) such that

\[
|f_{nk_n}| \leq \frac{1}{n^2 \lambda_n} e \quad \text{and} \quad |g_{nk_n}| \leq \frac{1}{n^2 \lambda_n} e.
\]

Define

\[
f = \sum_{n=1}^\infty \lambda_n f_{nk_n} \quad \text{and} \quad g = \sum_{n=1}^\infty \lambda_n g_{nk_n}.
\]
(e-uniformly convergent series). Since the terms in the series defining $f$ are mutually disjoint and since $T$ is disjointness preserving, it is easy to see that $0 \leq \lambda_n |Tf_{nk_n}| \leq |Tf|$ for all $n$. Similarly we get that $0 \leq \lambda_n |Tg_{nk_n}| \leq |Tg|$ $(n = 1, 2, \ldots)$. Therefore

$$0 \leq \lambda_n w_n \leq \lambda_n |Tf_{nk_n}| + \lambda_n |Tg_{nk_n}| \leq |Tf| + |Tg|$$

for all $n$. Hence $\{\lambda_n w_n\}_{n=1}^{\infty}$ is order bounded.

**Theorem 8.** Let $L$ and $M$ be Archimedean Riesz spaces such that $L$ is uniformly complete and for each disjoint sequence $0 < w_n \in M$ $(n = 1, 2, \ldots)$ there exist positive real numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $\{\lambda_n w_n\}_{n=1}^{\infty}$ is not order bounded in $M$. Then each disjointness preserving operator $T$ from $L$ into $M$ is order bounded on some order dense ideal in $L$.

**Proof.** Let $A_T$ be the largest ideal in $L$ on which $T$ is order bounded. We claim that $A_T = \{0\}$. Indeed, suppose $A_T \neq \{0\}$ and take $0 < e \in A_T$. Since the ideal generated by $e$ is not finite dimensional, there exist mutually disjoint elements $0 < e_n \leq e$ $(n = 1, 2, \ldots)$. Since $T$ is not order bounded on the ideal generated by $e_n$, there exists $0 < u_n \leq e_n$ such that $w_n = (|Tu_n| - |Te_n|)^+ > 0$ $(n = 1, 2, \ldots)$. Now it follows from Lemma 7 that $\{\lambda_n w_n\}_{n=1}^{\infty}$ is order bounded for any sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive real numbers, which contradicts the assumption on $M$. Hence $A_T = \{0\}$, i.e., $A_T^d = L$.

The above theorem has the following immediate corollary.

**Corollary 9.** If $L$ is a Banach lattice and $M$ is a normed Riesz space, then any disjointness preserving operator from $L$ into $M$ is order bounded on some order dense ideal in $L$.

It follows from a combination of Proposition 6 and Theorem 8 that if $L$ is a uniformly complete Riesz space with the property that for each disjoint sequence $0 < w_n \in L$ $(n = 1, 2, \ldots)$ there exist positive real numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $\{\lambda_n w_n\}_{n=1}^{\infty}$ is not order bounded, then any band preserving operator in $L$ is an orthomorphism. This implies in particular that any band preserving operator on a Banach lattice is order bounded, and hence norm bounded (as any order bounded operator on a Banach lattice is norm bounded), which is a result of Yu. A. Abramovich, A. I. Veksler and A. V. Koldunov [2, 3]. We note, however, that using Lemma 7 it is not difficult to prove the following result, which is due to A. W. Wickstead (presented at the meeting on Riesz spaces and operator theory in Oberwolfach, 1982): If $L$ is a uniformly complete Riesz space with the property that for each $0 < e \in L$ such that the order interval $[0, e]$ does not contain any atoms, there exists a disjoint sequence $\{e_n\}_{n=1}^{\infty}$ in $[0, e]$ such that $\{\lambda_n e_n\}_{n=1}^{\infty}$ is not bounded in $L$ for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive real numbers, then each band preserving operator in $L$ is automatically order bounded. The case when $L$ is in addition Dedekind $\sigma$-complete was proved by W. A. J. Luxemburg [8, Theorem 9.9].

The condition that every disjointness preserving operator from an Archimedean Riesz space $L$ into itself is order bounded is quite strong. Indeed, let $P$ be a prime ideal in $L$ which is not uniformly closed. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in $P$
and \( f_0 \in L \setminus P \) such that \( f_n \to f_0 \) (r.u.). Let \( \varphi \) be any linear functional on \( L \) with \( \varphi(f) = 0 \) for all \( f \in P \) and \( \varphi(f_0) = 1 \). Take \( 0 < e \in L \) and define the disjointness preserving operator \( T \) by \( Tf = \varphi(f)e \) for all \( f \in L \). Since \( f_n \to f_0 \) (r.u.), \( T f_n = 0 \) \((n = 1, 2, \ldots)\) and \( T f_0 = e > 0 \), \( T \) is not order bounded. Furthermore we note that a proper prime ideal \( P \) in \( L \) is uniformly closed iff \( P \) is a maximal ideal (e.g., this follows from a combination of \([9, \text{Theorems 27.1, 33.2 and 60.2}]\)). Therefore, if every disjointness preserving operator in \( L \) is order bounded, then every proper prime ideal in \( L \) is a maximal ideal. By \([9, \text{Theorem 37.6}]\), this last condition is equivalent to the property that the quotient Riesz space \( L/A \) is Archimedean for every ideal \( A \) in \( L \). Riesz spaces with this property are called hyper-Archimedean. We thus have the following result.

**Theorem 10.** If \( L \) is an Archimedean Riesz space such that any disjointness preserving operator in \( L \) is order bounded, then \( L \) is hyper-Archimedean.

Finally, we note that if \( L \) is uniformly complete and hyper-Archimedean, then every principal ideal in \( L \) is finite dimensional \([9, \text{Theorem 61.4}]\) and hence any linear operator on \( L \) is order bounded.

**References**


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