BANACH SPACES THAT HAVE NORMAL STRUCTURE AND ARE ISOMORPHIC TO A HILBERT SPACE

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Abstract. We prove that given a Hilbert space \((E, \| \cdot \|)\), and \(\| \cdot \|\) a norm on \(E\) such that for all \(x \in E\), \(1/\beta x \leq \|x\| \leq |x|\) for some \(\beta\), if \(1 \leq \beta < \sqrt{2}\), then \((E, \| \cdot \|)\) satisfies a convexity property from which normal structure follows.

1. Introduction. A Banach space \(E\) is said to have normal structure if for each bounded, closed and convex subset \(C\) of \(E\), consisting of more than one point, there is an \(x \in C\) such that

\[
\sup \{ \|x - y\| : y \in C \} < \text{diam}(C) = \sup \{ \|y_1 - y_2\| : y_1, y_2 \in C \}.
\]

In [4] it was proved that if \(E\) has normal structure, \(C \subseteq E\) is a nonempty weakly compact convex set, and \(T : C \to C\) is a mapping such that for all \(x, y \in C\), \(\|Tx - Ty\| \leq \|x - y\|\), then \(T\) has a fixed point in \(C\).

For \(r > 1\) let \(E_r\) be the space \(l_2\) renormed by

\[
\|x\|_r := \max\{ \|x\|_2, r\|x\|_\infty \},
\]

where \(\| \cdot \|_2\) and \(\| \cdot \|_\infty\) denote the \(l_2\) and \(l_\infty\) norms, respectively. It is known from [1] that \(E_r\) has normal structure when \(r < \sqrt{2}\).

We use the idea of multidimensional volumes and their related convexity moduli to prove \(E_r\) satisfies a convexity property that implies this result. The notion of volumes in Banach spaces was introduced by Silverman and its use in defining moduli of convexity was introduced in [5]. Roughly speaking, the modulus of \(k\)-rotundity, \(\delta_k(\varepsilon)\), measures the depth below the surface of the unit sphere of the centroid of a simplex of \(k + 1\) norm-1 vectors enclosing a \(k\)-dimensional volume larger than \(\varepsilon\). In symbols,

\[
A(x_1, \ldots, x_{k + 1}) \geq \varepsilon
\]

implies that

\[
\|(x_1 + \cdots + x_{k + 1})/(k + 1)\| \leq 1 - \delta_k(\varepsilon).
\]

Here \(A(x_1, \ldots, x_{k + 1})\) denotes the enclosed volume. In case \(k = 1\), \(A(x_1, x_2) = \|x_1 - x_2\|\) and \(\delta_1(\varepsilon)\) is the usual modulus of convexity. In all cases

\[
D(\| \cdot \|, x_1, \ldots, x_{k + 1}) \leq A(x_1, \ldots, x_{k + 1}).
\]
Here

$$D(\|\cdot\|, x_1, \ldots, x_{k+1}) = \|x_k - x_{k+1}\| \cdot \text{dist}(x_{k-1}, [x_k, x_{k+1}])$$

$$\cdots \cdot \text{dist}(x_1, [x_2, \ldots, x_{k+1}])$$

where $[x_{i+1}, \ldots, x_{k+1}]$ is the affine span of the vectors $x_{i+1}, \ldots, x_{k+1}$ and

$$\text{dist}(x_i, [x_{i+1}, \ldots, x_{k+1}]) = \inf\{\|x_i - x\|: x \in [x_{i+1}, \ldots, x_{k+1}]\}.$$ 

For a Hilbert space the inequality is always equality.

A connection between these moduli and normal structure of a Banach space $E$ was given in [3], namely

**Lemma.** Suppose that for some $\delta > 0$ and some $0 < \varepsilon < 1$ there is an integer $m$ such that for all norm-1 $x_1, \ldots, x_m \in E$, if $\|(x_1 + x_2 + \cdots + x_m)/m\| > 1 - \delta$ then $D(\|\cdot\|, x_1, \ldots, x_m) < \varepsilon$.

Then $E$ is super-reflexive and has normal structure.

**2. The result.**

**Theorem.** Let $(E, \|\cdot\|)$ be a Hilbert space, and let $\|\cdot\|$ be a norm on $E$ such that for all $x \in E$, $1/\beta|x| \leq \|x\| \leq |x|$ for some $\beta$, $1 < \beta < \sqrt{2}$. Given $\varepsilon > 0$, there exists $\delta > 0$ and $M$, a positive integer, such that for $m \geq M$, if $x_1, \ldots, x_m \in E$, $|x_1|, \ldots, |x_m| \leq 1$, and $\|(x_1 + \cdots + x_m)/m\|^2 > 1 - \delta$, then $D(\|\cdot\|, x_1, \ldots, x_m) < \varepsilon$.

The proof requires some preliminary results.

**Lemma 1.** Given $k$, a positive integer and $\alpha > 0$, let $f$, $g$ be the functions from $\mathbb{R}^k$ into $\mathbb{R}$ defined by

$$f(x_1, \ldots, x_k) = \frac{1}{k + 1} \sum_{i=1}^k \frac{i}{i + 1} (x_{k+1-i})^2,$$

$$g(x_1, \ldots, x_k) = \alpha - \prod_{i=1}^k x_i, \quad (x_1, \ldots, x_k) \in \mathbb{R}^k.$$

Then $f(x) \geq (k/(k + 1)) (\alpha^{2/k}/(k + 1)^{1/k})$ whenever $x \in \mathbb{R}^k$ and $g(x) = 0$.

**Proof.** Let $w \in \Omega = \{x \in \mathbb{R}^k: g(x) = 0\}$ and $y = f(w) > 0$. Then $\hat{\Omega} \equiv f^{-1}([0, y]) \cap \Omega$ is nonempty and compact and, thus, there exists $x^* \in \hat{\Omega}$, a global minimum point of $f$ over $\hat{\Omega}$. Also, if $z \in \Omega \setminus f^{-1}([0, y])$ then $f(z) > y$ so that $x^*$ is a global minimum point of $f$ over $\Omega$.

With $x^* = (b_1, \ldots, b_k)$ then $\prod_{i=1}^k b_i = \beta > 0$. Thus $\nabla g(x^*) \neq 0$, where $\nabla g$ is the gradient of $g$. It now follows, by Lagrange's theorem, that for some $\lambda \in \mathbb{R}$,

$$\nabla f(x^*) = \lambda \nabla g(x^*).$$

So, for each $i$, $1 \leq i \leq k$,

$$\frac{2}{k + 1} \frac{k + 1 - i}{k + 2 - i} b_i = \lambda \prod_{j=1}^k b_j.$$
or
\[
\frac{2}{k+1} \frac{k+1-i}{k+2-i} b_i^2 - \lambda \beta = 0.
\]

Thus, for each \(1 \leq i \leq k\), \(((k + 1)/2)\lambda \beta = ((k + 1 - i)/(k + 2 - i))b_i^2\), and
\[
\left[ \frac{k + 1 - i}{k+2-i} b_i^2 \right] = \prod_{j=1}^{i} \frac{k + 1 - j}{k+2-j} b_j^2 = \frac{1}{k+1} \beta^2.
\]

Therefore, \(((k + 1 - i)/(k + 2 - i))b_i^2 = \beta^{2/k}/(k + 1)^{1/k}\) for each \(1 \leq i \leq k\).

So, given \(x \in \Omega\),
\[
f(x) > f(x^*) = \frac{1}{k+1} \sum_{i=1}^{k} \frac{i}{i+1} b_{k+1-i}^2
\]
\[
= \frac{1}{k+1} \sum_{i=1}^{k} \frac{k+1-i}{k+2-i} b_i^2 = \frac{1}{k+1} \frac{\beta^{2/k}}{(k+1)^{1/k}}.
\]

**Lemma 2.** Let \((E, \| \cdot \|)\) be a Hilbert space and \(k\) any positive integer. Given \(x_1, \ldots, x_{k+1} \in E, \|x_1\|, \ldots, \|x_{k+1}\| < 1\), then
\[
\frac{1}{k+1} \frac{\sum_{i=1}^{k} i}{k+1} \|x_k+1\| < 1 - \frac{1}{k+1} \sum_{i=1}^{k} \frac{i}{i+1} \|x_{k+1-i} - \frac{\sum_{j=k+2-i}^{k+1} x_j}{i}\|^2.
\]

**Proof.** Since \((E, \| \cdot \|)\) is a Hilbert space, given \(x, y \in E\), then
\[
\frac{1}{k+1} \frac{k}{k+1} \|y\|^2 = \frac{1}{k+1} \|x\|^2 + \frac{k}{(k+1)^2} \|x - y\|^2.
\]

In particular, given \(x_1, \ldots, x_{k+1} \in E\),
\[
\frac{1}{k+1} \frac{\sum_{i=1}^{k} i}{k+1} \|x_{k+1}\|^2 = \frac{1}{k+1} \|x_1\|^2 + \frac{k}{(k+1)^2} \left(\|x_2 + \cdots + x_{k+1}\|^2 - \frac{k}{(k+1)^2} \|x_1 - \frac{x_2 + \cdots + x_{k+1}}{k}\|^2\right).
\]

The proof of the lemma now follows by induction on \(k\).

**Lemma 3.** Let \((E, \| \cdot \|)\) be a Hilbert space and \(k\) a positive integer. If \(x_1, \ldots, x_{k+1} \in E, \|x_1\|, \ldots, \|x_{k+1}\| \leq 1, D(\| \cdot \|, x_1, \ldots, x_{k+1}) \geq \varepsilon > 0\), then
\[
\|(x_1 + \cdots + x_{k+1})/(k + 1)\|^2 \leq 1 - (k/(k+1))(\varepsilon^{2/k}/(k+1)^{1/k}).
\]

**Proof.** Since \(D(\| \cdot \|, x_1, \ldots, x_{k+1}) \geq \varepsilon\) then \(D(\| \cdot \|, x_1, \ldots, x_{k+1}) = \beta\), where \(\beta \geq \varepsilon\). Let \(d_i = \text{dist}(x_i, [x_{i+1}, \ldots, x_{k+1}])\), for each \(1 \leq i \leq k\). By Lemma 2, with \(f\) as defined in Lemma 1, it follows that
\[
\frac{1}{k+1} \frac{\sum_{i=1}^{k} i}{k+1} \|x_{k+1}\|^2 \leq 1 - \frac{1}{k+1} \sum_{i=1}^{k} \frac{i}{i+1} \|x_{k+1-i} - \frac{\sum_{j=k+2-i}^{k+1} x_j}{i}\|^2
\]
\[
\leq 1 - \frac{1}{k+1} \sum_{i=1}^{k} \frac{i}{i+1} \frac{d_{k+1-i}^2}{i} = 1 - f(d_1, \ldots, d_k).
\]
However, $\prod_{i=1}^{k} d_i = D(\| \cdot \|, x_1, \ldots, x_{k+1}) = \beta$. So, by Lemma 1,

$$f(d_1, \ldots, d_k) \geq \left(\frac{k}{k+1}\right)\left(\frac{\beta^{2/k}}{(k+1)^{1/k}}\right),$$

and, therefore,

$$\|(x_1 + \cdots + x_{k+1})/(k+1)\|^2 \leq 1 - \left(\frac{k}{k+1}\right)\left(\frac{\epsilon^{2/k}}{(k+1)^{1/k}}\right).$$

**Remark.** Extending these ideas [2] gives the exact value of the modulus of $k$-rotundity of a Hilbert space, e.g.

$$\delta_k(\epsilon) = 1 - \left[1 - \frac{k}{k+1} \frac{\epsilon^{2/k}}{(k+1)^{1/k}}\right]^{1/2}. $$

**Proof of the Theorem.** Choose $\eta > 0$ so that $\beta^2 + \eta < 2$. Given any $\epsilon > 0$, let $\beta > 0$ a positive integer, let $\Delta_k(\epsilon) = \left(\frac{k}{k+1}\right)\left(\frac{\epsilon^{2/k}}{(k+1)^{1/k}}\right)$. Since $\lim_{k \to \infty} \Delta_k(\epsilon) = 1$, select $M > 1$ so large that $\Delta_{m-1}(\epsilon) > 1 - \eta$ whenever $m \geq M$.

Now, let $\delta = 2 - \beta^2 - \eta$, and suppose $m \geq M$, $(x_1 + \cdots + x_m)/m^2 > 1 - \delta$, $|x_1|, \ldots, |x_m| \leq 1$, while $D(1 \cdot x_1, \ldots, x_m) \geq \epsilon$. Then $\|x_1\|, \ldots, |x_m| \leq 1$ and

$$D(1, x_1, \ldots, x_m) \geq (1/\beta)^{m-1} \cdot D(1, x_1, \ldots, x_m) \geq (1/\beta)^{m-1} \epsilon.$$ 

It follows from Lemma 3 that

$$\left\|\frac{x_1 + \cdots + x_m}{m}\right\|^2 \leq 1 - \Delta_{m-1}\left(\frac{1}{\beta}\right)^{m-1}\epsilon.$$ 

However,

$$1 - \delta \leq \left\|\frac{x_1 + \cdots + x_m}{m}\right\|^2 \leq \beta^2 \left\|\frac{x_1 + \cdots + x_m}{m}\right\|^2$$

$$\leq \beta^2 \left(1 - \Delta_{m-1}\left(\frac{1}{\beta}\right)^{m-1}\epsilon\right) = \beta^2 \left(1 - \left(\frac{1}{\beta}\right)^2 \Delta_{m-1}(\epsilon)\right)$$

$$= \beta^2 - \Delta_{m-1}(\epsilon) < \beta^2 + \eta - 1.$$ 

This contradicts the definition of $\delta$. Therefore, $D(1, x_1, \ldots, x_m) < \epsilon$. Q.E.D.

The result proven in [1] now follows from this theorem and the lemma mentioned in the introduction.

**Corollary.** Let $(E, \| \cdot \|)$ be a Hilbert space and let $| \cdot |$ be a norm on $E$ such that for all $x \in E$, $1/\beta |x| \leq \|x\| \leq |x|$ for some $\beta$, $1 \leq \beta < \sqrt{2}$. Then $(E, | \cdot |)$ has normal structure.

Note that the Theorem is sharp because Baillon and Schöneberg proved that $E_r$ fails to have normal structure for $r \geq \sqrt{2}$. 

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References


