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A TWO WEIGHT INEQUALITY FOR THE FRACTIONAL INTEGRAL WHEN $p = n/\alpha$

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ABSTRACT. Let I_α be the fractional integral operator defined as

$$I_\alpha f(x) = \int f(y)|x-y|^{\alpha-n}\,dy.$$

Given a weight w (resp. v), necessary and sufficient conditions are given for the existence of a nontrivial weight v (resp. w) such that

$$\|v\chi_B\|_\infty \frac{1}{|B|}\int_B |I_\alpha f(x) - m_B(I_\alpha f)|\,dx \leqslant C\left(\int |f|^{n/\alpha} w\right)^{\alpha/n}$$

holds for any ball B such that $\|v\chi_B\|_\infty > 0$.

1. Introduction. We consider the fractional integral operator I_α, $0<\alpha<n$, defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} f(y)|x-y|^{\alpha-n}\,dy. \tag{1.1}$$

Necessary and sufficient conditions were obtained in [1] in order that given a weight v (resp. w) there exists a nontrivial weight w (resp. v) satisfying

$$\left(\int_{\mathbf{R}^n} |I_\alpha f(x)|^q v(x)\,dx\right)^{1/q} \leqslant \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}$$

for $1<p$, $q<\infty$, $1/q \geqslant 1/p - \alpha/n$. For the case $p=1$, $q = n/(n-\alpha)$ weights satisfying a weak type inequality were characterized. Our purpose now is to study the limiting case $p = n/\alpha$, $q=\infty$.

It is not difficult to verify that, except for trivial cases, I_α is not a bounded operator from $L^{n/\alpha}(w\,dx)$ into $L^\infty(v\,dx)$. To see this we assume the set $\{x\colon v(x)>0\} \cap \{x\colon w(x)<\infty\}$ has positive Lebesgue measure. Then if B_1 is the unit ball we may assume that for some N the set $G = \{x\colon v(x)>0\}\cap\{x\colon w(x)<N\}\cap B_1$ has positive measure and zero as a point of density. Take $f(y) = \chi_G(y)|y|^{-\beta}$, with $\beta<\alpha$. Then

$$\int |f|^{n/\alpha} w\,dy \leqslant N\int_{B_1} |y|^{-\beta n/\alpha}\,dy \leqslant \frac{N\omega_n\alpha}{n(\alpha-\beta)}.$$

On the other hand, since $I_\alpha f(x)$ is continuous at zero, we have

$$\|I_\alpha f\|_{L^\infty(v)} \geqslant I_\alpha f(0) = \int_G |y|^{-\beta}|y|^{\alpha-n}\,dy \geqslant \int_{G\cap B_r} |y|^{\alpha-\beta-n}\,dy,$$

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where r is such that $|B_s \cap G|/|B_s| \geqslant 3/4$, for every $s \leqslant r$. We write

$$A_k = \{y\colon r4^{-(k+1)/n} \leqslant |y| < r4^{-k/n}\}$$

and

$$C_k = \{y\colon 2^{-1/n}r4^{-k/n} \leqslant |y| < r4^{-k/n}\}.$$

Then C_k is contained in A_k and

$$|G \cap A_k| \geqslant 2\omega_n r^n 4^{-(k+1)} = |C_k|.$$

Taking into account that $|y|^{\alpha-\beta-n}$ is a decreasing function, we have

$$\int_{G\cap B_r} |y|^{\alpha-\beta-n}dy = \sum_{k=0}^{\infty} \int_{G\cap A_k} |y|^{\alpha-\beta-n}dy \geqslant \sum_{k=0}^{\infty} \int_{C_k} |y|^{\alpha-\beta-n}dy$$
$$= \omega_n \cdot \frac{r^{\alpha-\beta}}{\alpha-\beta} \cdot \frac{1}{(1+2^{(\beta-\alpha)/n})}.$$

Therefore, if $\|I_\alpha f\|_{L^\infty(v)} \leqslant C\|f\|_{L^{n/\alpha}(w)}$ were true, we would have

$$r^{\alpha-\beta} \leqslant C(2^{(\beta-\alpha)/n}+1)(\alpha-\beta)^{1-\alpha/n}$$

for any $\beta < \alpha$. Letting β go to α, we arrive at a contradiction.

Moreover, as is well known, the function $f(x) = (|x|^\alpha \log|x|)^{-1}\chi_{(2,\infty)}(|x|)$ belongs to $L^{n/\alpha}(dx)$, yet the integral (1.1) defining $I_\alpha f(x)$ is divergent for every x.

However, if f belongs to $L^{n/\alpha}(dx)$ and has compact support, $I_\alpha f(x)$ is finite for almost every x. Furthermore, given any ball $B = B(z,r)$ the expression

$$I_\alpha^B f(x) = \int_B f(y)|x-y|^{\alpha-n}dy + \int_{CB} f(y)[|x-y|^{\alpha-n} - |y-z|^{\alpha-n}]\,dy$$

is well defined for every f in $L^{n/\alpha}(dx)$ and coincides almost everywhere (a.e.) with $I_\alpha f$ up to a finite constant $C_B = \int_{CB} f(y)|y-z|^{\alpha-n}dy$, if in addition, f has compact support.

These observations lead us to study, as in **[2]**, the weights satisfying the substitute inequality

$$\|v\chi_B\|_\infty \frac{1}{|B|}\int_B |I_\alpha f(x) - m_B(I_\alpha f)|dx \leqslant \left(\int |f|^{n/\alpha} w\,dx\right)^{\alpha/n}, \tag{1.2}$$

for any ball B such that $\|v\chi_B\|_\infty > 0$ and f with compact support. We are using the notation $|E|$ to indicate the Lebesgue measure of the set E and $m_E(g)$ the average of g over E, i.e. $m_E(g) = (1/|E|)\int_E g(y)\,dy$.

2. The results. We begin by studying those weights w for which (1.2) holds for some nontrivial weight v. We first prove the following

LEMMA 1. *Let v and g be measurable functions satisfying*

$$\|v\chi_S\|_\infty \frac{1}{|S|}\int_S |g - m_S(g)| \leqslant C \tag{2.1}$$

for any ball S such that $\|v\chi_S\|_\infty > 0$. Then if B and B^* are two balls such that

$|B| = |B^*|$ *and* $\|v\chi_B\|_\infty > 0$, *we have*

$$\|v\chi_B\|_\infty \frac{1}{|B|}\int_B |g - m_{B^*}(g)| \leqslant 3C\frac{|\tilde{B}|}{|B|}$$

where $\tilde{B}$ *is any ball containing* $B \cup B^*$.

PROOF.

$$\|v\chi_B\|_\infty \frac{1}{|B|}\int_B |g - m_{B^*}(g)|$$

$$\leqslant \|v\chi_B\|_\infty \left[\frac{1}{|B|}\int_B |g - m_B(g)| + |m_B(g) - m_{\tilde{B}}(g)| + |m_{B^*}(g) - m_{\tilde{B}}(g)|\right]$$

$$\leqslant C + \|v\chi_{\tilde{B}}\|_\infty \left[\frac{1}{|B|}\int_B |g - m_{\tilde{B}}(g)| + \frac{1}{|B^*|}\int_{B^*} |g - m_{\tilde{B}}(g)|\right]$$

$$\leqslant C + 2\frac{|\tilde{B}|}{|B|}\frac{1}{|\tilde{B}|}\int_{\tilde{B}} |g - m_{\tilde{B}}(g)| \leqslant 3\frac{|\tilde{B}|}{|B|}C.$$

From this lemma we can easily obtain a necessary condition on the weight w for (1.2) to hold.

THEOREM 1. *Let* w *be a nonnegative function, finite on a set of positive measure and such that there exists a nonnegative function* v, *not identically zero, satisfying* (1.2) *for any bounded function with compact support. Then, for any* R *large enough, we have*

$$\int_{|x|\leqslant R} w(x)^{-\alpha/(n-\alpha)}\,dx \leqslant CR^n. \tag{2.2}$$

PROOF. Let $w_\varepsilon(x) = w(x) + \varepsilon$ and define $f_R = w_\varepsilon^{-\alpha/(n-\alpha)}\chi_{B_R}$ for R large enough so that $\|v\chi_{B_R}\|_\infty > 0$. Then f_R is a bounded function with compact support and

$$\int |f_R|^{n/\alpha} w = \int_{B_R} w_\varepsilon^{-n/(n-\alpha)} w \leqslant \int_{B_R} w_\varepsilon^{-\alpha/(n-\alpha)} < \infty.$$

Let us take $B_R^* = B(z,R)$, the ball centered at z of radius R, with $|z| = 5R$. Clearly B_R and B_R^* are contained in $\tilde{B}_R = B(0,6R)$ and $K = |\tilde{B}_R|/|B_R|$ is independent of R. Also, substituting f_R for f in (1.2) we obtain that $g_R = I_\alpha(f_R)$ satisfies (2.1) with a constant $C_R = (\int_{B_R} w_\varepsilon^{-\alpha/(n-\alpha)})^{\alpha/n}$. Hence, we can apply Lemma 1 to conclude

$$\|v\chi_{B_R}\|_\infty \frac{1}{|B_R|}\int_{B_R} |g_R - m_{B_R^*}(g_R)| \leqslant 3K\left(\int_{B_R} w_\varepsilon^{-\alpha/(n-\alpha)}\right)^{\alpha/n}.$$

Now for $x \in B_R$ we have

$$g_R(x) - m_{B_R^*}(g_R) = \frac{1}{|B_R^*|}\int_{B_R^*}\int_{B_R} f_R(y)[|x-y|^{\alpha-n} - |t-y|^{\alpha-n}]\,dy\,dt$$

$$\geqslant \frac{1}{|B_R^*|}\int_{B_R^*}\int_{B_R} f_R(y)[(2R)^{\alpha-n} - (3R)^{\alpha-n}]\,dy\,dt$$

$$\geqslant CR^{\alpha-n}\int_{B_R} w_\varepsilon^{-\alpha/(n-\alpha)}\,dy$$

with $C > 0$ and independent of R. Therefore, since we can always assume $\|v\chi_{B_R}\|_\infty \geqslant 1$ for R large enough, we obtain

$$R^{\alpha-n}\int_{B_R} w_\varepsilon^{-\alpha/(n-\alpha)} \leqslant C\left(\int_{B_R} w_\varepsilon^{-\alpha/(n-\alpha)}\right)^{\alpha/n},$$

which implies, for R large enough,

$$\int_{B_R} w_\varepsilon^{-\alpha/(n-\alpha)} \leqslant CR^n.$$

Now letting ε go to zero we obtain the desired conclusion. □

We now want to study the behavior of the fractional integral operator acting on functions of $L^{n/\alpha}(w\,dx)$ for a weight w satisfying (2.2). As in the case of Lebesgue measure, we can show that if $w^{-\alpha/(n-\alpha)}$ is merely locally integrable, the integral defining $I_\alpha f$ is finite almost everywhere for any $f \in L^{n/\alpha}(w\,dx)$ having compact support. In fact, if $B = B(0,R)$ is a ball containing the support of f and $f \geqslant 0$, we have

$$\begin{aligned}\int_B\int f(y)|x-y|^{\alpha-n}\,dy\,dx &= \int f(y)\int_B |x-y|^{\alpha-n}\,dx\,dy \leqslant \int f(y)\int_{B(y,2R)}|x-y|^{\alpha-n}\,dx\,dy\\ &\leqslant CR^\alpha\left(\int f^{n/\alpha}w\right)^{\alpha/n}\left(\int_B w^{-\alpha/(n-\alpha)}\right)^{1-\alpha/n} < \infty.\end{aligned}$$

Therefore $I_\alpha f$ is finite a.e.

The next theorem shows that condition (2.2) on w allows us to construct a weight v satisfying (1.2).

THEOREM 2. *Let w be a nonnegative function, finite on a set of positive measure, satisfying* (2.2) *for $R \geqslant 1$. Then there exists a nonnegative function v, not identically zero, such that* (1.2) *holds for any ball B satisfying $\|v\chi_B\|_\infty > 0$, and for any function f with compact support.*

PROOF. Let the maximal function be denoted by

$$M^*g(x) = \sup\left\{\frac{1}{|B(z,r)|}\int_{B(z,r)}|g(y)|\,dy \colon x \in B(z,r),\ 0 < r \leqslant 2\right\}.$$

Since $w^{-\alpha/(n-\alpha)}$ is a locally integrable function, $M^*(w^{-\alpha/(n-\alpha)})$ is finite a.e. We may assume that for N large enough the set $E = B(0,1) \cap \{x\colon M^*(w^{-\alpha/(n-\alpha)})(x) < N\}$ has positive measure. We claim that the weight $v = \chi_E$ satisfies (1.2).

Let f be a function in $L^{n/\alpha}(w\,dx)$ with compact support. In order to prove (1.2) we need only consider balls B such that $B \cap E \neq \varnothing$. If $B = B(z,R)$ is one of those balls, denoting by $\tilde{B}$ the ball $B(z,4R)$, we write

$$I_\alpha f(x) = I_\alpha^1 f(x) + I_\alpha^2 f(x) = \int_{\tilde{B}} f(y)|x-y|^{\alpha-n}\,dy + \int_{C\tilde{B}} f(y)|x-y|^{\alpha-n}\,dy.$$

For $I_\alpha^1 f$ we have

$$\begin{aligned}\frac{1}{|B|}\int_B |I_\alpha^1(f)(x) - m_B(I_\alpha^1 f)|\,dx &\leqslant \frac{2}{|B|}\int_B\int_{\tilde{B}}|f(y)||x-y|^{\alpha-n}\,dy\,dx\\ &\leqslant \frac{2}{|B|}\int_{\tilde{B}}|f(y)|\int_{B(y,5R)}|x-y|^{\alpha-n}\,dx\,dy\\ &\leqslant CR^{\alpha-n}\left(\int|f|^{n/\alpha}w\right)^{\alpha/n}\left(\int_{\tilde{B}}w^{-\alpha/(n-\alpha)}\right)^{1-\alpha/n}.\end{aligned}$$

If $4R \geqslant 1$, since $E \cap B \neq \varnothing$, it follows that $\tilde{B} \subset B(0,9R)$ and, therefore, by hypothesis

$$\int_{\tilde{B}} w^{-\alpha/(n-\alpha)} \leqslant CR^n.$$

On the other hand, if $4R \leqslant 1$ and $t \in E \cap B$, we get

$$\int_{\tilde{B}} w^{-\alpha/(n-\alpha)} \leqslant CR^n M^*(w^{-\alpha/(n-\alpha)})(t) \leqslant CNR^n.$$

So, in any case, we obtain

$$\frac{1}{|B|}\int_B |I_\alpha^1 f(x) - m_B(I_\alpha^1 f)|\,dx \leqslant C\left(\int|f|^{n/\alpha}w\right)^{\alpha/n}. \tag{2.3}$$

We now estimate $I_\alpha^2 f$:

$$\int_B |I_\alpha^2 f(x) - m_B(I_\alpha^2 f)|\,dx \leqslant \frac{1}{|B|}\int_B\int_B\int_{C\tilde{B}}|f(y)|\,\big||x-y|^{\alpha-n} - |t-y|^{\alpha-n}\big|\,dy\,dt\,dx.$$

But, using the mean value theorem and the fact that $||x-y| - |t-y|| < 2R$ for x and t in B and y in $C\tilde{B}$, it follows that

$$\big||x-y|^{\alpha-n} - |t-y|^{\alpha-n}\big| \leqslant CR|z-y|^{\alpha-n-1}.$$

Therefore

$$\begin{aligned}\frac{1}{|B|}\int_B |I_\alpha^2 f(x) - m_B(I_\alpha^2 f)|\,dx &\leqslant CR\int_{C\tilde{B}}|f(y)|\,|z-y|^{\alpha-n-1}\,dy\\ &\leqslant CR\left(\int|f|^{n/\alpha}w\right)^{\alpha/n}\left(\int_{C\tilde{B}}w(y)^{-\alpha/(n-\alpha)}|z-y|^{-n\beta}\,dy\right)^{1-\alpha/n},\end{aligned} \tag{2.4}$$

where $\beta = 1 + 1/(n-\alpha) > 1$. For the last integral we have

$$I = \int_{|z-y|\geqslant 4R} w(y)^{-\alpha/(n-\alpha)}|z-y|^{-n\beta}\,dy \leqslant \sum_{k=0}^{\infty}(2^kR)^{-n\beta}\int_{|z-y|\leqslant 2^{k+1}R} w(y)^{-\alpha/(n-\alpha)}\,dy.$$

If $|z| \geqslant 2$, since $B \cap E \neq \varnothing$, we have $R \geqslant |z|/2 \geqslant 1$ and, hence,

$$\int_{|z-y|\leqslant 2^{k+1}R} w(y)^{-\alpha/(n-\alpha)}\,dy \leqslant \int_{|y|\leqslant 2^{k+2}R} w(y)^{-\alpha/(n-\alpha)}\,dy \leqslant C(2^kR)^n.$$

Moreover, if $|z| \leqslant 2$ but k is such that $2^k R \geqslant 1$, the last estimate also holds. On the other hand, if $2^k R \leqslant 1$ and $t \in E \cap B$, we obtain

$$\int_{|z-y| \leqslant 2^{k+1}R} w(y)^{-\alpha/(n-\alpha)}\,dy \leqslant C(2^k R)^n M^*(w^{-\alpha/(n-\alpha)})(t) \leqslant CN(2^k R)^n.$$

Therefore

$$I \leqslant CR^{-n/(n-\alpha)} \sum_{k=0}^{\infty} 2^{-kn/(n-\alpha)} \leqslant CR^{-n/(n-\alpha)}.$$

Replacing this estimate in (2.4) gives

$$\frac{1}{|B|}\int_B |I_\alpha^2 f(x) - m_B(I_\alpha^2 f)|\,dx \leqslant C\left(\int |f|^{n/\alpha} w\right)^{\alpha/n}. \tag{2.5}$$

Taking into account that $\|v\|_\infty = 1$, the estimates (2.3) and (2.5) prove the claim. □

Extension of I_α to the whole space $L^{n/\alpha}(w\,dx)$. Let w be a weight satisfying (2.2). As we have seen, the integral (1.1), defining the fractional integral $I_\alpha f$, is absolutely convergent for any function f in $L^{n/\alpha}(w\,dx)$ with compact support. Let v be a weight satisfying (1.2). The previous theorem shows there always exists such a v. Then I_α can be considered as a bounded operator from a dense subspace of $L^{n/\alpha}(w\,dx)$ into a weighted version of BMO, denoted BMO(v). The norm on this space is given by

$$|||g||| = \sup_B \|\chi_B v\|_\infty m_B(|g - m_B(g)|),$$

where the sup is taken over the balls B such that $\|\chi_B v\|_\infty > 0$. Therefore I_α can be extended as a bounded operator from $L^{n/\alpha}(w\,dx)$ into BMO(v).

Furthermore, by arguments similar to those used in the proof of Theorem 2, it is possible to give an explicit expression for $I_\alpha f$ as an element in the space BMO(v), valid for any function f in $L^{n/\alpha}(w\,dx)$. In order to do this, assume w satisfies (2.2) for $R \geqslant 1$. For any $r > 0$ we define

$$I_r f(x) = \int_{|y|<r} f(y)|x-y|^{\alpha-n}\,dy + \int_{|y|\geqslant r} f(y)(|x-y|^{\alpha-n} - |y|^{\alpha-n})\,dy.$$

Let us show that for any f in $L^{\alpha/n}(w\,dx)$ this expression is finite a.e. For any R large enough we can write

$$I_r f(x) = I_\alpha(f\chi_{B_R})(x) + \int_{|y|\geqslant R} f(y)(|x-y|^{\alpha-n} - |y|^{\alpha-n})\,dy$$
$$- \int_{r\leqslant |y| < R} f(y)|y|^{\alpha-n}\,dy.$$

By the assumption on f and w, the last integral is a finite constant. Moreover, for any x such that $2|x| < R$, we have

$$\left|\int_{|y|\geqslant R} f(y)(|x-y|^{\alpha-n} - |y|^{\alpha-n})\,dy\right| \leqslant CR\int_{|y|\geqslant R} |f(y)|\,|y|^{\alpha-n-1}\,dy$$
$$\leqslant C\|f\|_{L^{n/\alpha}(w)}\left(\int_{|y|\geqslant R} w(y)^{-\alpha/(n-\alpha)}|y|^{-n\beta}\,dy\right)^{1-\alpha/n},$$

with $\beta = 1 + 1/(n-\alpha)$. Proceeding as in the proof of Theorem 2 we see that the last integral is finite. This proves our assertion. Moreover, we have also shown that $I_R f$ and $I_r f$ coincide a.e. up to a finite constant.

From these remarks we can conclude that for any $r > 0$ and any f in $L^{n/\alpha}(w\,dx)$, the function $I_r f$ coincides in BMO(v) with $I_\alpha(f)$ defined by density arguments, providing the expression we were looking for. □

We now consider the problem of characterizing those weights v for which there exists a nontrivial weight w satisfying (1.2).

THEOREM 3. *Let v be a nonnegative function different from zero on a set of positive measure. Then there exists a nonnegative function w finite on a set of positive measure and satisfying* (1.2) *for any bounded function f with compact support if and only if the function v satisfies* $|v(x)| \leqslant C(1+|x|)^{n-\alpha}$.

PROOF. Assume (1.2) holds for some w. Let $f(x) = \chi_E(x)$, where

$$E = B(0,1) \cap \{x\colon w(x) < N\},$$

for N large enough. By using translations if necessary, we can assume $|E| > 0$. Let $B = B(0,R)$ with $R \geqslant 1$ and large enough so that $\|v\chi_B\|_\infty > 0$. Let B^* be the ball $B(z,R)$ where z is such that $|z| = 5R$, and let $\tilde{B}$ be the ball centered at zero with radius $6R$. Therefore, if (1.2) is satisfied, we can apply Lemma 1 to $g = I_\alpha f$ and obtain

$$\|v\chi_B\|_\infty \frac{1}{|B|}\int_B |I_\alpha f(y) - m_{B^*}(I_\alpha f)|\,dy \leqslant K$$

for a constant K independent of R. Proceeding now as in the proof of Theorem 1 we obtain that, for any R large enough, $\|v\chi_B\|_\infty \leqslant CR^{n-\alpha}$, which implies

$$|v(x)| \leqslant C(1+|x|)^{n-\alpha} \quad \text{a.e.}$$

Conversely, we will show that (1.2) holds for $v(x) = (1+|x|)^{n-\alpha}$ and $w(x) = (1+|x|)^{(n+\varepsilon)(n-\alpha)/\alpha}$. Let $B = B(z,R)$ be any ball and $\tilde{B} = B(z,4R)$. As in the proof of Theorem 2 we write

$$I_\alpha f(x) = I_\alpha^1 f(x) + I_\alpha^2 f(x) = \int_{\tilde{B}} f(y)|x-y|^{\alpha-n}\,dy + \int_{C\tilde{B}} f(y)|x-y|^{\alpha-n}\,dy$$

for a bounded function f with compact support. We have already seen that for a function of this sort we have the estimate

$$\frac{1}{|B|}\int_B |I_\alpha^1 f(x) - m_B(I_\alpha^1 f)|\,dx \leqslant C\left(R^{-n}\int_{\tilde{B}} w^{-\alpha/(n-\alpha)}\right)^{1-\alpha/n}\left(\int |f|^{n/\alpha} w\right)^{\alpha/n}.$$

Consider

$$A(z,R) = (1+|z|+R)^{n-\alpha}\left(R^{-n}\int_{\tilde{B}} w^{-\alpha/(n-\alpha)}\right)^{1-\alpha/n}.$$

We want to show it is bounded independently of z and R. From our choice of w it follows that

$$M(w^{-\alpha/(n-\alpha)})(x) \leqslant C(1+|x|)^{-n},$$

where M is the usual Hardy-Littlewood maximal function operator. In particular $R^{-n}\int_{\tilde{B}} w^{-\alpha/(n-\alpha)} \leqslant C$. Thus, we need only consider $|z| + R \geqslant 1$. Now, if $|z| \geqslant R$,

$$A(z,R) \leqslant C|z|^{n-\alpha}\left[M(w^{-\alpha/(n-\alpha)})(z)\right]^{1-\alpha/n} \leqslant C,$$

and if $|z| \leqslant R$,

$$A(z,R) \leqslant CR^{n-\alpha}\left(R^{-n}\int w^{-\alpha/(n-\alpha)}\right)^{1-\alpha/n} \leqslant C.$$

Therefore

$$\|v\chi_B\|_\infty \frac{1}{|B|}\int_B |I_\alpha^1 f(x) - m_B(I_\alpha^1 f)|\,dx \leqslant CA(z,R)\left(\int |f|^{n/\alpha} w\right)^{\alpha/n} \leqslant C\left(\int |f|^{n/\alpha} w\right)^{\alpha/n}. \tag{2.6}$$

We also proved (see 2.4) that if $\beta = 1 + 1/(n-\alpha)$, then

$$\frac{1}{|B|}\int_B |I_\alpha^2 f(x) - m_B(I_\alpha^2 f)|\,dx \leqslant CR\left(\int |f|^{n/\alpha} w\right)^{\alpha/n}\left(\int_{|z-y|\geqslant 4R} w(y)^{-\alpha/(n-\alpha)}|z-y|^{-n\beta}\,dy\right)^{1-\alpha/n}.$$

From our choice of w we have the estimates

$$\begin{aligned}\int_{|z-y|\geqslant 4R} w(y)^{-\alpha/(n-\alpha)}|z-y|^{-n\beta}\,dy &\leqslant C\sum_{k=2}^{\infty}(2^k R)^{-n\beta}\int_{|z-y|<2^{k+1}R} w(y)^{-\alpha/(n-\alpha)}\,dy \\ &\leqslant CM(w^{-\alpha/(n-\alpha)})(z)R^{-n(\beta-1)}\sum_{k=2}^{\infty} 2^{n(1-\beta)k} \\ &\leqslant CR^{-n/(n-\alpha)}(1+|z|)^{-n}\end{aligned}$$

and

$$\begin{aligned}\int_{|z-y|\geqslant 4R} w(y)^{-\alpha/(n-\alpha)}|z-y|^{-n\beta}\,dy &\leqslant CR^{-n\beta}\int w(y)^{-\alpha/(n-\alpha)}\,dy \\ &\leqslant CR^{-n(n-\alpha+1)/(n-\alpha)}.\end{aligned}$$

Using these estimates for $|z| \geqslant R$ and $|z| \leqslant R$, respectively, we obtain

$$\|v\chi_B\|_\infty \frac{1}{|B|}\int_B |I_\alpha^2 f(x) - m_B(I_\alpha^2 f)|\,dx \leqslant C\left(\int |f|^{n/\alpha} w\right)^{\alpha/n}. \tag{2.7}$$

Combining (2.6) and (2.7), (1.2) follows. □

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