A TWO WEIGHT INEQUALITY FOR THE FRACTIONAL INTEGRAL WHEN $p = n/\alpha$

ELEONOR HARBOURE, ROBERTO A. MACÍAS AND CARLOS SEGOVIA

Abstract. Let $I_\alpha$ be the fractional integral operator defined as

$$I_\alpha f(x) = \int f(y)|x-y|^{\alpha-n}dy.$$ 

Given a weight $w$ (resp. $v$), necessary and sufficient conditions are given for the existence of a nontrivial weight $v$ (resp. $w$) such that

$$\|vX_B\|_{\infty} \frac{1}{|B|} \int_B |I_\alpha f(x) - m_B(I_\alpha f)|dx \leq C \left( \int |f|^{\alpha/n}w \right)^{\alpha/n}$$

holds for any ball $B$ such that $\|vX_B\|_{\infty} > 0$.

1. Introduction. We consider the fractional integral operator $I_\alpha$, $0 < \alpha < n$, defined by

$$(1.1) \quad I_\alpha f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n}dy.$$ 

Necessary and sufficient conditions were obtained in [1] in order that given a weight $v$ (resp. $w$) there exists a nontrivial weight $w$ (resp. $v$) satisfying

$$\left[ \int |I_\alpha f(x)|^q v(x)dx \right]^{1/q} \leq \left( \int |f(x)|^p w(x)dx \right)^{1/p}$$

for $1 < p, q < \infty$, $1/q > 1/p - \alpha/n$. For the case $p = 1$, $q = n/(n - \alpha)$ weights satisfying a weak type inequality were characterized. Our purpose now is to study the limiting case $p = n/\alpha$, $q = \infty$.

It is not difficult to verify that, except for trivial cases, $I_\alpha$ is not a bounded operator from $L^{n/\alpha}(wdx)$ into $L^\infty(vdx)$. To see this we assume the set $\{x: v(x) > 0\} \cap \{x: w(x) < \infty\}$ has positive Lebesgue measure. Then if $B_1$ is the unit ball we may assume that for some $N$ the set $G = \{x: v(x) > 0\} \cap \{x: w(x) < N\} \cap B_1$ has positive measure and zero as a point of density. Take $f(y) = \chi_G(y)|y|^{-\beta}$, with $\beta < \alpha$. Then

$$\int |f|^{\alpha/n}w dy \leq N \int_{B_1} |y|^{-\beta n/\alpha}dy \leq \frac{N\omega_n \alpha}{n(\alpha - \beta)}.$$ 

On the other hand, since $I_\alpha f(x)$ is continuous at zero, we have

$$\|I_\alpha f\|_{L^\infty(v)} \geq I_\alpha f(0) = \int_G |y|^{-\beta} |y|^{\alpha-n}dy \geq \int_{G \cap B_\epsilon} |y|^{\alpha-\beta-n}dy,$$
where \( r \) is such that \( |B_{i} \cap G|/|B_{i}| \geq 3/4 \), for every \( s \leq r \). We write
\[
A_{k} = \{ y : r4^{-(k+1)/n} \leq |y| < r4^{-k/n} \}
\]
and
\[
C_{k} = \{ y : 2^{-1/n}r4^{-k/n} \leq |y| < r4^{-k/n} \}.
\]
Then \( C_{k} \) is contained in \( A_{k} \) and
\[
|G \cap A_{k}| \geq 2 \omega_{n}r^{n}4^{-(k+1)} = |C_{k}|.
\]
Taking into account that \( |y|^\alpha \beta - n \) is a decreasing function, we have
\[
\int_{G \cap B_{r}} |y|^\alpha \beta - n \, dy = \sum_{k=0}^{\infty} \int_{G \cap A_{k}} |y|^\alpha \beta - n \, dy \geq \sum_{k=0}^{\infty} \int_{C_{k}} |y|^\alpha \beta - n \, dy
\]
\[
= \omega_{n} \cdot \frac{r^{\alpha - \beta}}{\alpha - \beta} \cdot \frac{1}{(1 + 2^{(\beta - \alpha)/n})}.
\]
Therefore, if \( \|I_{\alpha,f}\|_{L^{\infty}(v)} \leq C\|f\|_{L^{\infty}(w)} \) were true, we would have
\[
\frac{r^{\alpha - \beta}}{\alpha - \beta} \leq C(2^{(\beta - \alpha)/n} + 1)(\alpha - \beta)^{1 - \alpha/n}
\]
for any \( \beta < \alpha \). Letting \( \beta \) go to \( \alpha \), we arrive at a contradiction.

Moreover, as is well known, the function \( f(x) = (|x|^{\alpha} \log |x|)^{-1} \chi_{(2,\infty)}(|x|) \) belongs to \( L^{n/\alpha}(dx) \), yet the integral (1.1) defining \( I_{\alpha,f}(x) \) is divergent for every \( x \).

However, if \( f \) belongs to \( L^{n/\alpha}(dx) \) and has compact support, \( I_{\alpha,f}(x) \) is finite for almost every \( x \). Furthermore, given any ball \( B = B(z,r) \) the expression
\[
I_{\alpha}^{B,f}(x) = \int_{B} f(y)|x - y|^{\alpha - n} \, dy + \int_{CB} f(y)[|x - y|^{\alpha - n} - |y - z|^{\alpha - n}] \, dy
\]
is well defined for every \( f \) in \( L^{n/\alpha}(dx) \) and coincides almost everywhere (a.e.) with \( I_{\alpha}f \) up to a finite constant \( C_{B} = \int_{CB} f(y)|y - z|^{\alpha - n} \, dy \), if in addition, \( f \) has compact support.

These observations lead us to study, as in [2], the weights satisfying the substitute inequality
\[
(1.2) \quad \|v \chi_{B}\|_{\infty} \frac{1}{|B|} \int_{B} |I_{\alpha,f}(x)| \, dx \leq \left( \int |f|^{n/\alpha} w \, dx \right)^{\alpha/n},
\]
for any ball \( B \) such that \( \|v \chi_{B}\|_{\infty} > 0 \) and \( f \) with compact support. We are using the notation \( |E| \) to indicate the Lebesgue measure of the set \( E \) and \( m_{E}(g) \) the average of \( g \) over \( E \), i.e. \( m_{E}(g) = \frac{1}{|E|} \int_{E} g \, dy \).

2. The results. We begin by studying those weights \( w \) for which (1.2) holds for some nontrivial weight \( v \). We first prove the following

**Lemma 1.** Let \( v \) and \( g \) be measurable functions satisfying
\[
(2.1) \quad \|v \chi_{S}\|_{\infty} \frac{1}{|S|} \int_{S} |g - m_{S}(g)| \leq C
\]
for any ball \( S \) such that \( \|v \chi_{S}\|_{\infty} > 0 \). Then if \( B \) and \( B^{*} \) are two balls such that
$|B| = |B^*|$ and $\|v \chi_B\|_\infty > 0$, we have

$$\|v \chi_B\|_\infty \frac{1}{|B|} \int_B |g - m_{B^*}(g)| \leq 3C \frac{|\tilde{B}|}{|B|}$$

where $\tilde{B}$ is any ball containing $B \cup B^*$.

**Proof.**

$$\|v \chi_B\|_\infty \frac{1}{|B|} \int_B |g - m_{B^*}(g)| \leq \|v \chi_B\|_\infty \left[ \frac{1}{|B|} \int_B |g - m_B(g)| + |m_B(g) - m_{B^*}(g)| + |m_{B^*}(g) - m_{B^*}(g)| \right]$$

$$\leq C + \|v \chi_{B^*}\|_\infty \left[ \frac{1}{|B|} \int_B |g - m_{B^*}(g)| + \frac{1}{|B^*|} \int_B |g - m_{B^*}(g)| \right]$$

$$\leq C + 2 \frac{|\tilde{B}|}{|B|} \frac{1}{|B^*|} \int_B |g - m_{B^*}(g)| \leq 3 \frac{|\tilde{B}|}{|B|} C.$$  

From this lemma we can easily obtain a necessary condition on the weight $w$ for (1.2) to hold.

**Theorem 1.** Let $w$ be a nonnegative function, finite on a set of positive measure and such that there exists a nonnegative function $\nu$, not identically zero, satisfying (1.2) for any bounded function with compact support. Then, for any $R$ large enough, we have

$$\int \frac{w(x)^{-a/(n-a)}}{\nu(x)} dx \leq CR^n.$$  

**Proof.** Let $w(x) = w(x) + \epsilon$ and define $f_R = w^{-a/(n-a)} \chi_{B_R}$ for $R$ large enough so that $\|v \chi_{B_R}\|_\infty > 0$. Then $f_R$ is a bounded function with compact support and

$$\int f_R^{-a/(n-a)}w = \int_{B_R} w^{-n/(n-a)} \leq \int_{B_R} w^{-a/(n-a)} < \infty.$$  

Let us take $B_R^* = B(z, R)$, the ball centered at $z$ of radius $R$, with $|z| = 5R$. Clearly $B_R$ and $B_R^*$ are contained in $\tilde{B}_R = B(0, 6R)$ and $K = |\tilde{B}_R|/|B_R|$ is independent of $R$. Also, substituting $f_R$ for $f$ in (1.2) we obtain that $g_R = I_\alpha(f_R)$ satisfies (2.1) with a constant $C_R = (\int_{R} w^{-a/(n-a)})^{1/a}$. Hence, we can apply Lemma 1 to conclude

$$\|v \chi_{B_R}\|_\infty \frac{1}{|B_R|} \int_{B_R} |g_R - m_{B_R^*}(g_R)| \leq 3K \left( \int_{B_R} w^{-a/(n-a)} \right)^{1/a}.$$  

Now for $x \in B_R$ we have

$$g_R(x) - m_{B_R^*}(g_R) \geq \frac{1}{|B_R^*|} \int_{B_R^*} \int_{B_R} f_R(y) ||x - y||^{a-n} - ||t - y||^{a-n} dy dt$$

$$\geq \frac{1}{|B_R^*|} \int_{B_R^*} \int_{B_R} f_R(y) [(2R)^{a-n} - (3R)^{a-n}] dy dt$$

$$\geq CR^{a-n} \int_{B_R} w^{-a/(n-a)} dy.$$
with $C > 0$ and independent of $R$. Therefore, since we can always assume $\|v X_{B_R}\|_{\infty} \geq 1$ for $R$ large enough, we obtain

$$R^{a-n} \int_{B_R} w^{-a/(n-a)} \leq C \left( \int_{B_R} w^{-a/(n-a)} \right)^{a/n},$$

which implies, for $R$ large enough,

$$\int_{B_R} w^{-a/(n-a)} \leq CR^n.$$

Now letting $\varepsilon$ go to zero we obtain the desired conclusion. $\square$

We now want to study the behavior of the fractional integral operator acting on functions of $L^{n/a}(w \, dx)$ for a weight $w$ satisfying (2.2). As in the case of Lebesgue measure, we can show that if $w^{-a/(n-a)}$ is merely locally integrable, the integral defining $I_{a,f}$ is finite almost everywhere for any $f \in L^{n/a}(w \, dx)$ having compact support. In fact, if $B = B(0,R)$ is a ball containing the support of $f$ and $f \geq 0$, we have

$$\int_B \int f(y) |x-y|^{a-n} \, dy \, dx = \int_f(y) \int_B |x-y|^{a-n} \, dx \, dy \leq \int_f(y) \int_{B(y,2R)} |x-y|^{a-n} \, dx \, dy \leq CR^a \left( \int_B f^{n/a} w \right)^{a/n} \left( \int_B w^{-a/(n-a)} \right)^{1-a/n} < \infty.$$

Therefore $I_{a,f}$ is finite a.e.

The next theorem shows that condition (2.2) on $w$ allows us to construct a weight $v$ satisfying (1.2).

**Theorem 2.** Let $w$ be a nonnegative function, finite on a set of positive measure, satisfying (2.2) for $R \geq 1$. Then there exists a nonnegative function $v$, not identically zero, such that (1.2) holds for any ball $B$ satisfying $\|v X_B\|_{\infty} > 0$, and for any function $f$ with compact support.

**Proof.** Let the maximal function be denoted by

$$M^*g(x) = \sup \left\{ \frac{1}{|B(z,r)|} \int_{B(z,r)} |g(y)| \, dy : x \in B(z,r), 0 < r \leq 2 \right\}.$$

Since $w^{-a/(n-a)}$ is a locally integrable function, $M^*(w^{-a/(n-a)})$ is finite a.e. We may assume that for $N$ large enough the set $E = B(0,1) \cap \{x : M^*(w^{-a/(n-a)})(x) < N\}$ has positive measure. We claim that the weight $v = \chi_E$ satisfies (1.2).

Let $f$ be a function in $L^{n/a}(w \, dx)$ with compact support. In order to prove (1.2) we need only consider balls $B$ such that $B \cap E \neq \emptyset$. If $B = B(z,R)$ is one of those balls, denoting by $\tilde{B}$ the ball $B(z,4R)$, we write

$$I_{a,f}(x) = I^1_{a,f}(x) + I^2_{a,f}(x) = \int_B f(y) |x-y|^{a-n} \, dy + \int_{CB} f(y) |x-y|^{a-n} \, dy.$$
For $I_a^1 f$ we have
\[
\frac{1}{|B|} \int_B |I_a^1 f(x) - m_B(I_a^1 f)| dx \leq \frac{2}{|B|} \int_B \int_B |f(y)||x-y|^{\alpha-n} dy dx
\]
\[
\leq \frac{2}{|B|} \int_B |f(y)| \int_{B(y,5R)} |x-y|^{\alpha-n} dx dy
\]
\[
\leq CR^{\alpha-n} \left( \int |f|^{\alpha/n} w \right)^{\alpha/n} \left( \int_B w^{-\alpha/(n-\alpha)} \right)^{1-\alpha/n}.
\]
If $4R \geq 1$, since $E \cap B \neq \emptyset$, it follows that $\hat{B} \subset B(0,9R)$ and, therefore, by hypothesis
\[
\int_B w^{-\alpha/(n-\alpha)} \leq CR^n.
\]
On the other hand, if $4R \leq 1$ and $t \in E \cap B$, we get
\[
\int_B w^{-\alpha/(n-\alpha)} \leq CR^*M^*(w^{-\alpha/(n-\alpha)})(t) \leq CNR^n.
\]
So, in any case, we obtain
\[
(2.3) \quad \frac{1}{|B|} \int_B |I_a^1 f(x) - m_B(I_a^1 f)| dx \leq C \left( \int |f|^{\alpha/n} w \right)^{\alpha/n}.
\]
We now estimate $I_a^2 f$:
\[
\int_B |I_a^2 f(x) - m_B(I_a^2 f)| dx \leq \frac{1}{|B|} \int_B \int_B \int_{CB} |f(y)||x-y|^{\alpha-n} - |t-y|^{\alpha-n} dy dt dx.
\]
But, using the mean value theorem and the fact that $||x-y| - |t-y|| < 2R$ for $x$ and $t$ in $B$ and $y$ in $CB$, it follows that
\[
||x-y|^{\alpha-n} - |t-y|^{\alpha-n}| \leq CR|z - y|^{\alpha-n-1}.
\]
Therefore
\[
(2.4) \quad \frac{1}{|B|} \int_B |I_a^2 f(x) - m_B(I_a^2 f)| dx \leq CR \int_{CB} |f(y)||z-y|^{\alpha-n-1} dy
\]
\[
\leq CR \left( \int |f|^{\alpha/n} w \right)^{\alpha/n} \left( \int_{CB} w(y)^{-\alpha/(n-\alpha)}|z-y|^{-\beta} dy \right)^{1-\alpha/n},
\]
where $\beta = 1 + 1/(n-\alpha) > 1$. For the last integral we have
\[
I = \int_{|z-y| \geq 4R} w(y)^{-\alpha/(n-\alpha)}|z-y|^{-\beta} dy \leq \sum_{k=0}^{\infty} (2^k R)^{-\beta} \int_{|z-y| \leq 2^{k+1} R} w(y)^{-\alpha/(n-\alpha)} dy.
\]
If $|z| \geq 2$, since $B \cap E \neq \emptyset$, we have $R \geq |z|/2 \geq 1$ and, hence,
\[
\int_{|z-y| \leq 2^{k+1} R} w(y)^{-\alpha/(n-\alpha)} dy \leq \int_{|y| \leq 2^{k+2} R} w(y)^{-\alpha/(n-\alpha)} dy \leq C(2^k R)^n.
\]
Moreover, if $|z| \leq 2$ but $k$ is such that $2^k R \geq 1$, the last estimate also holds. On the other hand, if $2^k R \leq 1$ and $t \in E \cap B$, we obtain

$$\int_{|z-v| \leq 2^k R} w(y)^{-\alpha/(n-\alpha)} \, dy \leq \left(2^k R\right)^n M^*(w^{-\alpha/(n-\alpha)})(t) \leq CN(2^k R)^n.$$ 

Therefore

$$I \leq CR^{-n/(n-\alpha)} \sum_{k=0}^{\infty} 2^{-kn/(n-\alpha)} \leq CR^{-n/(n-\alpha)}.$$

Replacing this estimate in (2.4) gives

(2.5) $\frac{1}{|B|} \int_B |I_a^2 f(x) - m_B(I_a^2 f)| \, dx \leq C \left( \int |f|^{n/\alpha} w \right)^{\alpha/n}.$

Taking into account that $\|v\|_\infty = 1$, the estimates (2.3) and (2.5) prove the claim.

**Extension of $I_a$ to the whole space $L^{n/\alpha}(w \, dx)$.** Let $w$ be a weight satisfying (2.2). As we have seen, the integral (1.1), defining the fractional integral $I_a f$, is absolutely convergent for any function $f$ in $L^{n/\alpha}(w \, dx)$ with compact support. Let $v$ be a weight satisfying (1.2). The previous theorem shows there always exists such a $v$. Then $I_a$ can be considered as a bounded operator from a dense subspace of $L^{n/\alpha}(w \, dx)$ into a weighted version of BMO, denoted BMO($v$). The norm on this space is given by

$$\|g\|_B = \sup_B \|\chi_B v\|_\infty m_B (\|g - m_B(g)\|),$$

where the sup is taken over the balls $B$ such that $\|\chi_B v\|_\infty > 0$. Therefore $I_a$ can be extended as a bounded operator from $L^{n/\alpha}(w \, dx)$ into BMO($v$).

Furthermore, by arguments similar to those used in the proof of Theorem 2, it is possible to give an explicit expression for $I_a f$ as an element in the space BMO($v$), valid for any function $f$ in $L^{n/\alpha}(w \, dx)$. In order to do this, assume $w$ satisfies (2.2) for $R \geq 1$. For any $r > 0$ we define

$$I_r f(x) = \int_{|y| < r} f(y)|x - y|^{-\alpha - n} \, dy + \int_{|y| \geq r} f(y)((|x - y|^{\alpha - n} - |y|^{\alpha - n}) \, dy.$$

Let us show that for any $f$ in $L^{\alpha/n}(w \, dx)$ this expression is finite a.e. For any $R$ large enough we can write

$$I_r f(x) = I_a(f \chi_{B_r})(x) + \int_{|y| \geq R} f(y)((|x - y|^{\alpha - n} - |y|^{\alpha - n}) \, dy - \int_{|y| < R} f(y)|y|^{\alpha - n} \, dy.$$

By the assumption on $f$ and $w$, the last integral is a finite constant. Moreover, for any $x$ such that $2|x| < R$, we have

$$\left| \int_{|y| \geq R} f(y)((|x - y|^{\alpha - n} - |y|^{\alpha - n}) \, dy \right| \leq CR \int_{|y| \geq R} |f(y)| |y|^{\alpha - n - 1} \, dy \leq C\|f\|_{L^{\alpha/n}(w)} \left( \int_{|y| \geq R} w(y)^{-\alpha/n} \, dy \right)^{\alpha/n}.$$
with $\beta = 1 + 1/(n - \alpha)$. Proceeding as in the proof of Theorem 2 we see that the last integral is finite. This proves our assertion. Moreover, we have also shown that $I_R f$ and $I_f$ coincide a.e. up to a finite constant.

From these remarks we can conclude that for any $r > 0$ and any $f$ in $L^{n/\alpha}(\nu dx)$, the function $I_r f$ coincides in $\text{BMO}(\nu)$ with $I_{\alpha}(f)$ defined by density arguments, providing the expression we were looking for. \(\square\)

We now consider the problem of characterizing those weights $\nu$ for which there exists a nontrivial weight $\nu$ satisfying (1.2).

**Theorem 3.** Let $\nu$ be a nonnegative function different from zero on a set of positive measure. Then there exists a nonnegative function $\nu$ finite on a set of positive measure and satisfying (1.2) for any bounded function $f$ with compact support if and only if the function $\nu$ satisfies $|\nu(x)| \leq C(1 + |x|)^{n - \alpha}$.

**Proof.** Assume (1.2) holds for some $\nu$. Let $f(x) = \chi_E(x)$, where

$$E = B(0,1) \cap \{x : \nu(x) < N\},$$

for $N$ large enough. By using translations if necessary, we can assume $|E| > 0$. Let $B = B(0,R)$ with $R \geq 1$ and large enough so that $\|\nu\chi_B\|_\infty > 0$. Let $B^*$ be the ball $B(z,R)$ where $z$ is such that $|z| = 5R$, and let $\hat{B}$ be the ball centered at zero with radius $6R$. Therefore, if (1.2) is satisfied, we can apply Lemma 1 to $g = I_{\alpha} f$ and obtain

$$\|\nu\chi_B\|_\infty \leq \frac{1}{|B|} \int_B |I_{\alpha} f(y) - m_B(I_{\alpha} f)| dy \leq K$$

for a constant $K$ independent of $R$. Proceeding now as in the proof of Theorem 1 we obtain that, for any $R$ large enough, $\|\nu\chi_B\|_\infty \leq CR^{n-\alpha}$, which implies $|\nu(x)| \leq C(1 + |x|)^{n-\alpha}$ a.e.

Conversely, we will show that (1.2) holds for $\nu(x) = (1 + |x|)^{n-\alpha}$ and $\nu(x) = (1 + |x|)^{(n+\alpha)(n-\alpha)/\alpha}$. Let $B = B(z,R)$ be any ball and $\hat{B} = B(z,4R)$. As in the proof of Theorem 2 we write

$$I_{\alpha} f(x) = I_{\alpha}^1 f(x) + I_{\alpha}^2 f(x) = \int_B f(y) |x-y|^{\alpha-n} dy + \int_{CB} f(y) |x-y|^{\alpha-n} dy$$

for a bounded function $f$ with compact support. We have already seen that for a function of this sort we have the estimate

$$\frac{1}{|B|} \int_B |I_{\alpha} f(x) - m_B(I_{\alpha} f)| dx \leq C \left( R^{-n} \int_B w^{-\alpha/(n-\alpha)} \right)^{1-\alpha/n} \left( \int f^\alpha w \right)^{\alpha/n}.$$  

Consider

$$A(z,R) = (1 + |z| + R)^{n-\alpha} \left( R^{-n} \int_B w^{-\alpha/(n-\alpha)} \right)^{1-\alpha/n}.$$  

We want to show it is bounded independently of $z$ and $R$. From our choice of $\nu$ it follows that

$$M \left( w^{-\alpha/(n-\alpha)} \right)(x) \leq C(1 + |x|)^{-n},$$
where $M$ is the usual Hardy-Littlewood maximal function operator. In particular $R^{-n} \int_B w^{-\alpha/(n-\alpha)} \lesssim C$. Thus, we need only consider $|z| + R \geq 1$. Now, if $|z| \geq R$,

$$A(z, R) \leq C|z|^{\alpha} \left[ M(w^{-\alpha/(n-\alpha)})(z) \right]^{1-\alpha/n} \leq C,$$

and if $|z| \leq R$,

$$A(z, R) \leq CR^{\alpha-\alpha} \left( R^{-n} \int w^{-\alpha/(n-\alpha)} \right)^{1-\alpha/n} \leq C.$$

Therefore

$$(2.6) \quad \|v\chi_B\|_{\infty} \frac{1}{|B|} \int_B \left| I^1_a f(x) - m_B(I^1_a f) \right| dx \leq CA(z, R) \left( \int |f|^{n/\alpha} w \right)^{\alpha/n}$$

$$\leq C \left( \int |f|^{n/\alpha} w \right)^{\alpha/n}.$$

We also proved (see 2.4) that if $\beta = 1 + 1/(n - \alpha)$, then

$$\frac{1}{|B|} \int_B \left| I^2_a f(x) - m_B(I^2_a f) \right| dx \leq CR \left( \int |f|^{n/\alpha} w \right)^{\alpha/n} \left( \int_{|z-y| \geq 4R} w(y)^{-\alpha/(n-\alpha)} |z-y|^{-n\beta} dy \right)^{1-\alpha/n}.$$

From our choice of $w$ we have the estimates

$$\int_{|z-y| \geq 4R} w(y)^{-\alpha/(n-\alpha)} |z-y|^{-n\beta} dy \leq C \sum_{k=2}^{\infty} (2^k R)^{-n\beta} \int_{|z-y| < 2^{k+1} R} w(y)^{-\alpha/(n-\alpha)} dy$$

$$\leq CM \left( w^{\alpha/(n-\alpha)}(z) R^{-n(\beta-1)} \sum_{k=2}^{\infty} 2^{n(1-\beta)k} \right)$$

$$\leq CR^{-n/(n-\alpha)}(1 + |z|)^{-n}$$

and

$$\int_{|z-y| \geq 4R} w(y)^{-\alpha/(n-\alpha)} |z-y|^{-n\beta} dy \leq CR^{-n} \int w(y)^{-\alpha/(n-\alpha)} dy$$

$$\leq CR^{-n(n-\alpha+1)/(n-\alpha)}.$$

Using these estimates for $|z| \geq R$ and $|z| \leq R$, respectively, we obtain

$$(2.7) \quad \|v\chi_B\|_{\infty} \frac{1}{|B|} \int_B \left| I^2_a f(x) - m_B(I^2_a f) \right| dx \leq C \left( \int |f|^{n/\alpha} w \right)^{\alpha/n}.$$

Combining (2.6) and (2.7), (1.2) follows. □

REFERENCES


PROGRAMA ESPECIAL DE MATEMATICA APLICADA, CONICET, GÜEMES 3450, CC91, 3000 SANTA FE, ARGENTINA (Current address of E. Harboure and R. Macias)

FACULTAD DE CIENCIAS EXACTAS, FISICAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES, 1428–BUENOS AIRES, ARGENTINA (Current address of C. Segovia)