A CHARACTERIZATION OF SPECTRAL OPERATORS
ON HILBERT SPACES

KÔTARÔ TANAHASHI AND TAKASHI YOSHINO

Abstract. In [8] Wadhwa shows that if a bounded linear operator $T$ on a complex Hilbert space $H$ is a decomposable operator and has the condition (I), then $T$ is a spectral operator with a normal scalar part. In this paper, by using this result, we show that a weak decomposable operator $T$ is a spectral operator with a normal scalar part if and only if $T$ satisfies the assertion that (1) $T$ has the conditions (C) and (I) or that (2) every spectral maximal space of $T$ reduces $T$. This result improves [1, 6 and 7]. From this result, we can get a characterization of spectral operators, but this result does not hold in complex Banach space (see Remark 2).

1. Preliminaries. Let $H$ be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on $H$. Let $\sigma(T)$ be the spectrum of $T$ and $\text{Lat}(T)$ be the family of all invariant subspaces of $T$. $Y \in \text{Lat}(T)$ is called a spectral maximal space of $T$ if $Y$ contains all $Z \in \text{Lat}(T)$ such that $\sigma(T \mid Z) \subset \sigma(T \mid Y)$. Let $\text{SM}(T)$ be the family of all spectral maximal spaces of $T$. $T \in B(H)$ is called a decomposable (resp., weak decomposable) operator if for every finite open covering $\{G_1, \ldots, G_n\}$ of $\sigma(T)$, there exists a system $\{Y_1, \ldots, Y_n\}$ in $\text{SM}(T)$ such that (1) $H = Y_1 + \cdots + Y_n$ (resp., $H = \overline{Y_1} + \cdots + \overline{Y_n}$ where $\overline{Y}$ denotes the closure of $Y \subset H$) and (2) $\sigma(T \mid Y_i) \subset G_i$ for $i = 1, \ldots, n$. $T \in B(H)$ is said to have the single valued extension property or the condition (A) if there exists no nonzero analytic function $f$ such that $(z - T)f(z) \equiv 0$. If $T \in B(H)$ has the condition (A), then for every $x \in H$, there exists a maximal open set $\rho_T(x)$ in the complex plane $\mathbb{C}$ for which there exists the unique analytic function $x(z)$ such that $(z - T)x(z) \equiv x$ on $\rho_T(x)$. Let $\sigma_T(x) = \rho_T(x)'$ and $H_T(E) = \{x \in H \mid \sigma_T(x) \subset E\}$ for a subset $E$ of $\mathbb{C}$. $T \in B(H)$ with (A) is said to have the condition (B) if there exists $K > 0$ such that $\|x\| \leq K\|x + y\|$ for all $x$ and $y$ in $H$ with $\sigma_T(x) \cap \sigma_T(y) = \emptyset$. $T$ with (A) is said to have the condition (C) if $H_T(F)$ is closed for all closed sets $F$ in $\mathbb{C}$. $T \in B(H)$ with (A) and (C) is said to have the condition (I) if $\sigma_T(P_F x) \subset \sigma_T(x)$ for all closed sets $F$ in $\mathbb{C}$ and for all $x \in H$ where $P_F$ is the orthogonal projection of $H$ onto $H_T(F)$.

It is known that weak decomposable operators have the condition (A) and that decomposable operators have the conditions (A) and (C) (see [2 and 5]). And if $T \in B(H)$ has the conditions (A) and (C), then $H_T(F) \in \text{SM}(T)$ and $\sigma(T \mid H_T(F)) \subset F$ for all closed sets $F$ in $\mathbb{C}$ (see [2]).

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2. Main results.

**Lemma 1.** If $T \in B(H)$ is a weak decomposable operator and if every spectral maximal space of $T$ reduces $T$, then $T$ has the condition (C).

**Proof.** Let $F$ be a closed set in $C$, and $G$ be any open set containing $F$. Since $\{F^c, G\}$ is an open covering of $\sigma(T)$, there exist $Y_1$ and $Y_2$ in $SM(T)$ such that $H = Y_1 + Y_2$, $\sigma(T|Y_1) \subset F^c$ and $\sigma(T|Y_2) \subset G$. If $x \in H_T(F)$, there exist $x_n^i \in Y_i$ for $i = 1, 2$ such that $x = \lim_{n \to \infty}(x_n^1 + x_n^2)$. Let $P$ be the orthogonal projection of $H$ onto $Y_1$. Then we have $PT = TP$, hence $\sigma_T(Px) \subset \sigma_T(y) \cap \sigma(T|Y_1) \subset F \cap F^c = \emptyset$. This implies $Px = 0$. Hence $0 = Px = \lim_{n \to \infty}(Px_n^1 + Px_n^2) = \lim_{n \to \infty}(x_n^1 + x_n^2)$, and so

$$x = \lim_{n \to \infty}(x_n^1 + x_n^2) - \lim_{n \to \infty}(x_n^1 + Px_n^2) = \lim_{n \to \infty}(x_n^2 - Px_n^2).$$

Since spectral maximal spaces are hyperinvariant (see [2, Proposition 1.3.2]), we have $x \in Y_2$. Hence $H_T(F) \subset Y_2 \subset H_T(\sigma(T|Y_2)) \subset H_T(G)$. Since $G$ is any open set containing $F$, we have $H_T(F) \subset \cap Y_2 \subset \cap H_T(G) = H_T(\cap G) = H_T(F)$. Thus $H_T(F) = \cap Y_2$ is closed.

**Lemma 2.** Let $T \in B(H)$ be a weak decomposable operator with (B) and let $H_T(F)$ reduce $T$ for all closed sets $F$ in $C$. Then $T$ has the condition (C).

**Proof.** Let $F$ be a closed set in $C$, and $G$ any open set containing $F$. Since $\{F^c, G\}$ is an open covering of $\sigma(T)$, there exist $Y_1$ and $Y_2$ in $SM(T)$ such that $H = Y_1 + Y_2$, $\sigma(T|Y_1) \subset F^c$ and $\sigma(T|Y_2) \subset G$. If $x \in H_T(F)$, there exist $x_n^i \in Y_i$ for $i = 1, 2$ such that $x = \lim_{n \to \infty}(x_n^1 + x_n^2)$. Let $P$ be the orthogonal projection of $H$ onto $H_T(F)$. Then $PT = TP$ and $x = Px = \lim_{n \to \infty}(Px_n^1 + Px_n^2)$. Since $Y_1$ is hyperinvariant, we have $Px_n^1 \in Y_1 \cap H_T(F) \subset H_T(F^c) \cap H_T(F)$. If $y \in H_T(F^c) \cap H_T(F)$, then $y = \lim_{n \to \infty}y_n$ where $y_n \in H_T(F)$ and $\sigma_T(y) \cap \sigma_T(-y) \subset F^c \cap F = \emptyset$. Hence $y = 0$ because $\|y\| \leq K\|y - y_n\| \to 0$ $(n \to 0)$ by the condition (B). Hence $Px_n^1 = 0$, and so $x = \lim_{n \to \infty}Px_n^2 \in Y_2$ because $Y_2$ is hyperinvariant. Thus $H_T(F) \subset Y_2 \subset H_T(G)$. The rest of the proof is similar to the proof of Lemma 1.

**Lemma 3.** Let $T \in B(H)$ be a weak decomposable operator. If $T$ has the conditions (C) and (I), then $T$ is a decomposable operator.

**Proof.** We show $H = H_T(F) + H_T(\overline{F})$ for all closed sets $F$ in $C$. Then it is easy to show that $T$ is a decomposable operator. Since $H = H_T(F) \oplus H_T(F^c)$, we have only to show $H_T(F) \subset H_T(F^c)$. Let $G$ be any open set containing $F$. Since $\{F^c, G\}$ is an open covering of $\sigma(T)$, there exist $Y_1$ and $Y_2$ in $SM(T)$ such that $H = Y_1 + Y_2$, $\sigma(T|Y_1) \subset F^c$ and $\sigma(T|Y_2) \subset G$. If $x \in H_T(G)$, there exist $x_n^i \in Y_i$ for $i = 1, 2$ such that $x = \lim_{n \to \infty}(x_n^1 + x_n^2)$. Hence

$$0 = P_Gx = \lim_{n \to \infty}(P_Gx_n^1 + P_Gx_n^2) = \lim_{n \to \infty}(P_Gx_n^1 + x_n^2),$$

and so

$$x = \lim_{n \to \infty}(x_n^1 + x_n^2) - \lim_{n \to \infty}(P_Gx_n^1 + x_n^2) = \lim_{n \to \infty}(x_n^1 - P_Gx_n^1).$$
Since \( \sigma_T(PGx^n) \subset \sigma_T(x_n) \subset \sigma(T|Y) \subset F^c \), we have \( x_n^1 - P_Gx_n^1 \in H_T(F^c) \) and so \( x \in H_T(F^c) \). Hence \( H_T(G^+) \subset H_T(F^c) \), and so \( H_T(G^+) \supset H_T(F^c)^\perp \). Since \( G \) is any open set containing \( F \), we have \( H_T(F) = H_T(\cap \overline{G}) = \cap H_T(\overline{G}) \subset H_T(F^c)^\perp \). Thus \( H_T(F)^+ \subset H_T(F^c) \).

**Theorem.** If \( T \in B(H) \), then the following assertions are equivalent.

1. \( T = N + Q \) where \( N \in B(H) \) is a normal operator and \( Q \) is a quasinilpotent operator commuting with \( N \).
2. \( T \) is a weak decomposable operator with (C) and (I).
3. \( T \) is a weak decomposable operator and every spectral maximal space of \( T \) reduces \( T \).
4. \( T \) is a weak decomposable operator with (B), and \( H_T(F) \) reduces \( T \) for all closed set \( F \) in \( C \).

**Proof.** We show the implications \( (1) \Rightarrow (4) \Rightarrow (2) \Rightarrow (1) \) and \( (1) \Rightarrow (3) \Rightarrow (2) \).

(1) \Rightarrow (4) and (3). This implication is known. But we include it for completeness. (1) implies that \( T \) is a spectral operator with a normal scalar part \( N \). Hence \( T \) is a decomposable operator and \( T \) has the conditions (B) and (C). Hence \( T \) is a weak decomposable operator. Let \( E(\cdot) \) be the resolution of the identity of \( N \). Then \( H_T(F) = H_N(F) = E(F)H \) for all closed sets \( F \) in \( C \). This implies \( H_T(F) \) reduces \( T \). And if \( Y \in SM(T) \), then \( Y = HT(a(T\{ Y}) \), hence \( Y \) reduces \( T \) (see, for details, [2 and 3]).

(4) \Rightarrow (2). \( T \) has the condition (C) by Lemma 2. Since \( PF = TPF \) for all closed sets \( F \) in \( C \), \( T \) has the condition (I).

(2) \Rightarrow (1). Since \( T \) is a decomposable operator with (I) by Lemma 3, this implication follows from [8].

(3) \Rightarrow (2). \( T \) has the condition (C) by Lemma 1, hence \( H_T(F) \in SM(T) \) for all closed sets \( F \) in \( C \). Thus \( PF = TPF \) and \( T \) has the condition (I).

Since \( T \in B(H) \) is a spectral operator if and only if \( T \) is similar to some \( S \in B(H) \) which satisfies the condition (1) of the Theorem (see [3]), we have the following

**Corollary.** \( T \in B(H) \) is a spectral operator if and only if \( T \) is similar to some \( S \in B(H) \) which satisfies one of the conditions of the Theorem.

**Remark 1.** The condition (I) is introduced by Wadhwa in [8]. In [6] Jafarian proved the implication \( (2) \Rightarrow (1) \) of the Theorem under an assumption that \( T \) is reductive, i.e. every invariant subspace of \( T \) reduces \( T \).

**Remark 2.** Let \( X = L^\infty[0,1] \) be the Banach space of all essentially bounded complex valued functions on \([0,1]\) endowed with the essential supremum norm. Let \( T \in B(X) \) be the multiplication operator, i.e. \((Tx)(t) = tx(t)\) for \( t \in [0,1] \) and \( x \in X \). Then we can show that \( T \) is a decomposable operator and \( \sigma_T(x) = \text{ess supp } x \) by an argument similar to [4, p. 106]. \( X_T(F) \) stands for \( H_T(F) \). Define \( PF \in B(X) \) such that \((PFx)(t) = \chi_F(t)x(t)\) for \( t \in [0,1] \) where \( \chi_F(t) \) is the characteristic function of a closed set \( F \) in \( C \). Then \( PF \) is a projection of \( X \) onto \( X_T(F) \) (of course, \( PF \) is not selfadjoint) and \( PF = TPF \). Hence \( X_T(F) \) reduces \( T \), and \( T \) has the
condition (I). And if \( \sigma_T(x) \cap \sigma_T(y) = \emptyset \), then \(|x(t) + y(t)| \geq |x(t)|+|y(t)| \) a.e. \([0,1]\). Hence \( \|x+y\| \geq \|x\| \), thus \( T \) has the condition (B). But \( T \) is not a spectral operator. Because if \( T \) is spectral, then \( T \) has the resolution of the identity \( E(\cdot) \). Hence, \( x = \lim_{n \to \infty} x_n \) where \( x_n = E(\{0\} \cup \left[1/n,1\right])x \) for all \( x \in X \). Let \( x(t) \equiv 1 \). Since \( E(F)X = X_T(F) \) for all closed sets \( F \) in \( C \), \( \text{ess sup} x_n \subset \{0\} \cup \left[1/n,1\right] \). Hence \( \|x - x_n\| = 1 \) for all \( n \). This is a contradiction. Thus, the Theorem does not hold in this case.

**References**