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A CHARACTERIZATION OF SPECTRAL OPERATORS ON HILBERT SPACES

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ABSTRACT. In [8] Wadhwa shows that if a bounded linear operator T on a complex Hilbert space H is a decomposable operator and has the condition (I), then T is a spectral operator with a normal scalar part. In this paper, by using this result, we show that a weak decomposable operator T is a spectral operator with a normal scalar part if and only if T satisfies the assertion that (1) T has the conditions (C) and (I) or that (2) every spectral maximal space of T reduces T. This result improves [1, 6 and 7]. From this result, we can get a characterization of spectral operators, but this result does not hold in complex Banach space (see Remark 2).

1. Preliminaries. Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. Let $\sigma(T)$ be the spectrum of T and $\mathrm{Lat}(T)$ be the family of all invariant subspaces of T. $Y \in \mathrm{Lat}(T)$ is called a spectral maximal space of T if Y contains all $Z \in \mathrm{Lat}(T)$ such that $\sigma(T|Z) \subset \sigma(T|Y)$. Let $\mathrm{SM}(T)$ be the family of all spectral maximal spaces of T. $T \in B(H)$ is called a decomposable (resp., weak decomposable) operator if for every finite open covering $\{G_1, \dots, G_n\}$ of $\sigma(T)$, there exists a system $\{Y_1, \dots, Y_n\}$ in $\mathrm{SM}(T)$ such that (1) $H = Y_1 + \cdots + Y_n$ (resp., $H = \overline{Y_1 + \cdots + Y_n}$ where $\overline{Y}$ denotes the closure of $Y \subset H$) and (2) $\sigma(T|Y_i) \subset G_i$ for $i = 1, \dots, n$. $T \in B(H)$ is said to have the single valued extension property or the condition (A) if there exists no nonzero analytic function f such that $(z - T)f(z) \equiv 0$. If $T \in B(H)$ has the condition (A), then for every $x \in H$, there exists a maximal open set $\rho_T(x)$ in the complex plane $\mathbf{C}$ for which there exists the unique analytic function $x(z)$ such that $(z - T)x(z) \equiv x$ on $\rho_T(x)$. Let $\sigma_T(x) = \rho_T(x)^c$ and $H_T(E) = \{x \in H \mid \sigma_T(x) \subset E\}$ for a subset E of $\mathbf{C}$. $T \in B(H)$ with (A) is said to have the condition (B) if there exists $K > 0$ such that $\|x\| \leq K\|x + y\|$ for all x and y in H with $\sigma_T(x) \cap \sigma_T(y) = \varnothing$. T with (A) is said to have the condition (C) if $H_T(F)$ is closed for all closed sets F in $\mathbf{C}$. $T \in B(H)$ with (A) and (C) is said to have the condition (I) if $\sigma_T(P_F x) \subset \sigma_T(x)$ for all closed sets F in $\mathbf{C}$ and for all $x \in H$ where P_F is the orthogonal projection of H onto $H_T(F)$.

It is known that weak decomposable operators have the condition (A) and that decomposable operators have the conditions (A) and (C) (see [2 and 5]). And if $T \in B(H)$ has the conditions (A) and (C), then $H_T(F) \in \mathrm{SM}(T)$ and $\sigma(T|H_T(F)) \subset F$ for all closed sets F in $\mathbf{C}$ (see [2]).

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2. Main results.

LEMMA 1. *If $T \in B(H)$ is a weak decomposable operator and if every spectral maximal space of T reduces T, then T has the condition* (C).

PROOF. Let F be a closed set in $\mathbf{C}$, and G be any open set containing F. Since $\{F^c, G\}$ is an open covering of $\sigma(T)$, there exist Y_1 and Y_2 in $\mathrm{SM}(T)$ such that $H = \overline{Y_1 + Y_2}$, $\sigma(T | Y_1) \subset F^c$ and $\sigma(T | Y_2) \subset G$. If $x \in H_T(F)$, there exist $x_n^i \in Y_i$ for $i = 1, 2$ such that $x = \lim_{n \to \infty}(x_n^1 + x_n^2)$. Let P be the orthogonal projection of H onto Y_1. Then we have $PT = TP$, hence $\sigma_T(Px) \subset \sigma_T(x) \cap \sigma(T | Y_1) \subset F \cap F^c = \varnothing$. This implies $Px = 0$. Hence $0 = Px = \lim_{n \to \infty}(Px_n^1 + Px_n^2) = \lim_{n \to \infty}(x_n^1 + Px_n^2)$, and so

$$x = \lim_{n \to \infty}\left(x_n^1 + x_n^2\right) - \lim_{n \to \infty}\left(x_n^1 + Px_n^2\right) = \lim_{n \to \infty}\left(x_n^2 - Px_n^2\right).$$

Since spectral maximal spaces are hyperinvariant (see [**2**, Proposition 1.3.2]), we have $x \in Y_2$. Hence $H_T(F) \subset Y_2 \subset H_T(\sigma(T | Y_2)) \subset H_T(G)$. Since G is any open set containing F, we have $H_T(F) \subset \bigcap Y_2 \subset \bigcap H_T(G) = H_T(\bigcap G) = H_T(F)$. Thus $H_T(F) = \bigcap Y_2$ is closed.

LEMMA 2. *Let $T \in B(H)$ be a weak decomposable operator with* (B) *and let $\overline{H_T(F)}$ reduce T for all closed sets F in $\mathbf{C}$. Then T has the condition* (C).

PROOF. Let F be a closed set in $\mathbf{C}$, and G any open set containing F. Since $\{F^c, G\}$ is an open covering of $\sigma(T)$, there exist Y_1 and Y_2 in $\mathrm{SM}(T)$ such that $H = \overline{Y_1 + Y_2}$, $\sigma(T | Y_1) \subset F^c$ and $\sigma(T | Y_2) \subset G$. If $x \in H_T(F)$, there exist $x_n^i \in Y_i$ for $i = 1, 2$ such that $x = \lim_{n \to \infty}(x_n^1 + x_n^2)$. Let P be the orthogonal projection of H onto $\overline{H_T(F)}$. Then $PT = TP$ and $x = Px = \lim_{n \to \infty}(Px_n^1 + Px_n^2)$. Since Y_1 is hyperinvariant, we have $Px_n^1 \in Y_1 \cap \overline{H_T(F)} \subset H_T(F^c) \cap \overline{H_T(F)}$. If $y \in H_T(F^c) \cap \overline{H_T(F)}$, then $y = \lim_{n \to \infty} y_n$ where $y_n \in H_T(F)$ and $\sigma_T(y) \cap \sigma_T(-y_n) \subset F^c \cap F = \varnothing$. Hence $y = 0$ because $\|y\| \leqslant K\|y - y_n\| \to 0$ $(n \to 0)$ by the condition (B). Hence $Px_n^1 = 0$, and so $x = \lim_{n \to \infty} Px_n^2 \in Y_2$ because Y_2 is hyperinvariant. Thus $H_T(F) \subset Y_2 \subset H_T(G)$. The rest of the proof is similar to the proof of Lemma 1.

LEMMA 3. *Let $T \in B(H)$ be a weak decomposable operator. If T has the conditions* (C) *and* (I), *then T is a decomposable operator.*

PROOF. We show $H = H_T(F) + H_T(\overline{F^c})$ for all closed sets F in $\mathbf{C}$. Then it is easy to show that T is a decomposable operator. Since $H = H_T(F) \oplus H_T(F)^{\perp}$, we have only to show $H_T(F)^{\perp} \subset H_T(\overline{F^c})$. Let G be any open set containing F. Since $\{F^c, G\}$ is an open covering of $\sigma(T)$, there exist Y_1 and Y_2 in $\mathrm{SM}(T)$ such that $H = \overline{Y_1 + Y_2}$, $\sigma(T | Y_1) \subset F^c$ and $\sigma(T | Y_2) \subset G$. If $x \in H_T(\overline{G})^{\perp}$, there exist $x_n^i \in Y_i$ for $i = 1, 2$ such that $x = \lim_{n \to \infty}(x_n^1 + x_n^2)$. Hence

$$0 = P_{\overline{G}}x = \lim_{n \to \infty}\left(P_{\overline{G}}x_n^1 + P_{\overline{G}}x_n^2\right) = \lim_{n \to \infty}\left(P_{\overline{G}}x_n^1 + x_n^2\right),$$

and so

$$x = \lim_{n \to \infty}\left(x_n^1 + x_n^2\right) - \lim_{n \to \infty}\left(P_{\overline{G}}x_n^1 + x_n^2\right) = \lim_{n \to \infty}\left(x_n^1 - P_{\overline{G}}x_n^1\right).$$

Since $\sigma_T(P_{\bar{G}}x_n^1) \subset \sigma_T(x_n^1) \subset \sigma(T \mid Y_1) \subset F^c$, we have $x_n^1 - P_{\bar{G}}x_n^1 \in H_T(\overline{F^c})$ and so $x \in H_T(\overline{F^c})$. Hence $H_T(\bar{G})^\perp \subset H_T(\overline{F^c})$, and so $H_T(\bar{G}) \supset H_T(\overline{F^c})^\perp$. Since G is any open set containing F, we have $H_T(F) = H_T(\bigcap \bar{G}) = \bigcap H_T(\bar{G}) \supset H_T(\overline{F^c})^\perp$. Thus $H_T(F)^\perp \subset H_T(\overline{F^c})$.

THEOREM. *If $T \in B(H)$, then the following assertions are equivalent.*

(1) *$T = N + Q$ where $N \in B(H)$ is a normal operator and Q is a quasinilpotent operator commuting with N.*

(2) *T is a weak decomposable operator with* (C) *and* (I).

(3) *T is a weak decomposable operator and every spectral maximal space of T reduces T.*

(4) *T is a weak decomposable operator with* (B), *and $\overline{H_T(F)}$ reduces T for all closed set F in* $\mathbf{C}$.

PROOF. We show the implications (1) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (4) and (3). This implication is known. But we include it for completeness. (1) implies that T is a spectral operator with a normal scalar part N. Hence T is a decomposable operator and T has the conditions (B) and (C). Hence T is a weak decomposable operator. Let $E(\ )$ be the resolution of the identity of N. Then $H_T(F) = H_N(F) = E(F)H$ for all closed sets F in $\mathbf{C}$. This implies $H_T(F)$ reduces T. And if $Y \in \mathrm{SM}(T)$, then $Y = H_T(\sigma(T \mid Y))$, hence Y reduces T (see, for details, [**2** and **3**]).

(4) $\Rightarrow$ (2). T has the condition (C) by Lemma 2. Since $P_F T = T P_F$ for all closed sets F in $\mathbf{C}$, T has the condition (I).

(2) $\Rightarrow$ (1). Since T is a decomposable operator with (I) by Lemma 3, this implication follows from [**8**].

(3) $\Rightarrow$ (2). T has the condition (C) by Lemma 1, hence $H_T(F) \in \mathrm{SM}(T)$ for all closed sets F in $\mathbf{C}$. Thus $P_F T = T P_F$ and T has the condition (I).

Since $T \in B(H)$ is a spectral operator if and only if T is similar to some $S \in B(H)$ which satisfies the condition (1) of the Theorem (see [**3**]), we have the following

COROLLARY. *$T \in B(H)$ is a spectral operator if and only if T is similar to some $S \in B(H)$ which satisfies one of the conditions of the Theorem.*

REMARK 1. The condition (I) is introduced by Wadhwa in [**8**]. In [**6**] Jafarian proved the implication (2) $\Rightarrow$ (1) of the Theorem under an assumption that T is reductive, i.e. every invariant subspace of T reduces T.

REMARK 2. Let $X = L^\infty[0,1]$ be the Banach space of all essentially bounded complex valued functions on $[0,1]$ endowed with the essential supremum norm. Let $T \in B(X)$ be the multiplication operator, i.e. $(Tx)(t) = tx(t)$ for $t \in [0,1]$ and $x \in X$. Then we can show that T is a decomposable operator and $\sigma_T(x) = \operatorname{ess\,supp} x$ by an argument similar to [**4**, p. 106]. $X_T(F)$ stands for $H_T(F)$. Define $P_F \in B(X)$ such that $(P_F x)(t) = \chi_F(t)x(t)$ for $t \in [0,1]$ where $\chi_F(t)$ is the characteristic function of a closed set F in $\mathbf{C}$. Then P_F is a projection of X onto $X_T(F)$ (of course, P_F is not selfadjoint) and $P_F T = T P_F$. Hence $X_T(F)$ reduces T, and T has the condition (I). And if $\sigma_T(x) \cap \sigma_T(y) = \varnothing$, then $|x(t) + y(t)| = |x(t)| + |y(t)|$ a.e. $[0, 1]$. Hence $\|x + y\| \geqslant \|x\|$, thus T has the condition (B). But T is not a spectral operator. Because if T is spectral, then T has the resolution of the identity $E(\ )$. Hence, $x = \lim_{n \to \infty} x_n$ where $x_n = E(\{0\} \cup [1/n, 1])x$ for all $x \in X$. Let $x(t) \equiv 1$. Since $E(F)X = X_T(F)$ for all closed sets F in $\mathbf{C}$, $\operatorname{ess\,supp} x_n \subset \{0\} \cup [1/n, 1]$. Hence $\|x - x_n\| = 1$ for all n. This is a contradiction. Thus, the Theorem does not hold in this case.

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