PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 4, April 1984

THE VERY WELL POISED ${}_6\psi_6$. II

RICHARD ASKEY[1]

ABSTRACT. A proof of Bailey's sum of the very well poised ${}_6\psi_6$ series is obtained from a simple difference equation and special cases that are easy to evaluate.

1. Introduction. The very well poised ${}_6\psi_6$ series was summed by Bailey [7]. This is a very important identity and a number of proofs have been found [**7**, **10**, **1**, **5**]. All of these proofs used the special case of the very well poised ${}_6\phi_5$. A simple derivation of this sum from orthogonal polynomials was given in [**2**], and the last of the proofs mentioned above showed how to obtain the ${}_6\psi_6$ sum directly from the ${}_6\phi_5$ sum with minimal work. However, it is still annoying that a sum that is this important has not been obtained from a more elementary special case. Recently an integral that must have some relationship with the ${}_6\psi_6$ sum was evaluated via a functional equation and one simple case, so it is natural to see if the same type of argument will work on the ${}_6\psi_6$ series. It does, but it is a bit more complicated.

A basic hypergeometric series has the form Σc_n with c_{n+1}/c_n a rational function of q^n for a fixed number q. We will take $|q| < 1$. Ramanujan [**9**, p. 196, formula 17] stated the identity

$$(1.1)\qquad {}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix}; q, x\right) := \sum_{-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty (b/a;q)_\infty}{(x;q)_\infty (b/ax;q)_\infty (b;q)_\infty (q/a;q)_\infty}.$$

See [**8** and **3**] for simple proofs. Bailey [7] showed that

$$(1.2)\qquad {}_6\psi_6\left(\begin{matrix} aq, -aq, ab, ac, ad, ae \\ a, -a, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, \frac{q}{bcde}\right) = \Pi_q\left[\begin{matrix} a^2q, q, q/a^2, q/bc, q/bd, q/be, q/cd, q/ce, q/de \\ q/ab, q/ac, q/ad, q/ae, aq/b, aq/c, aq/d, aq/e, q/bcde \end{matrix}\right]$$

Received by the editors May 26, 1983.
1980 *Mathematics Subject Classification*. Primary 33A35.
Key words and phrases. Basic hypergeometric series.
[1]Supported in part by N.S.F. Grant MCS-8101568 and in part by the Graduate School of the University of Wisconsin-Madison.

where

$$(1.3)\qquad {}_p\psi_p\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_p\end{matrix};q,x\right):=\sum_{-\infty}^{\infty}\frac{(a_1;q)_n\cdots(a_p;q)_n}{(b_1;q)_n\cdots(b_p;q)_n}x^n,$$

$$(1.4)\qquad (a;q)_\infty:=\prod_{n=0}^{\infty}(1-aq^n),$$

$$(1.5)\qquad (a;q)_n:=(a;q)_\infty/(aq^n;q)_\infty,$$

$$(1.6)\qquad \Pi_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_p\end{matrix}\right]:=\frac{(a_1;q)_\infty\cdots(a_p;q)_\infty}{(b_1;q)_\infty\cdots(b_p;q)_\infty}.$$

The adjective well poised means the parameters in (1.3) can be paired so that the product of a numerator and denominator parameter is the same for each pair. In (1.2) the parameters are arranged in order and the product is a^2q in each case. The "very" part is used to denote a well poised series where two numerator parameters are q times the corresponding denominator parameters, and contribute

$$\frac{1-a^2q^{2n}}{1-a^2}$$

to the nth term of the series. This is not quite enough to determine the very well poised ${}_6\psi_6$ since nothing has been said about the power series variable. The appropriate power series variable is given in (1.2). If (1.2) is summed in the other order using

$$(a;q)_{-n}=(-1)^n q^{n(n+1)/2}a^{-n}/(q/a;q)_n,$$

then it follows that

$$(1.7)\qquad {}_6\psi_6\left(\begin{matrix}aq,\ -aq,ab,ac,ad,ae\\ a,\ -a,aq/b,aq/c,aq/d,aq/e\end{matrix};q,\frac{q}{bcde}\right)$$
$$={}_6\psi_6\left(\begin{matrix}q/a,\ -q/a,b/a,c/a,d/a,e/a\\ a^{-1},\ -a^{-1},q/ba,q/ca,q/da,q/ea\end{matrix};q,\frac{q}{bcde}\right).$$

Thus one reason for the choice of power series variable is that it is invariant under this change. The real reason is that it is the choice that allows the series to be summed.

From (1.7) the convergence condition is $|q/bcde|<1$, and none of the denominator parameters can be q^{-k}, $k=0,1,\ldots$. Thus all of the denominator parameters on the right-hand side of (1.2) are natural.

The integral that was evaluated in **[4]** (see **[6]** for the first evaluation which works for a slightly more general set of conditions on the parameters) is

$$(1.8)\qquad \int_{-1}^{1}\frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})\,dx}{h(x,a)h(x,b)h(x,c)h(x,d)\sqrt{1-x^2}}$$

where

$$h(x, a) = \prod_{n=0}^{\infty} (1 - 2axq^n + a^2q^{2n}) \tag{1.9}$$

and $\max(|a|, |b|, |c|, |d|, |q|) < 1$.

If $f(a, b, c, d)$ denotes this integral the trick is to consider

$$bf(aq, b, c, d) - af(a, bq, c, d),$$

so that the factor that is introduced to recover $f(a, b, c, d)$ is independent of x. We want to do the same with (1.2). However, it is better to rewrite it first as

$$\sum_{-\infty}^{\infty} \frac{(1 - a^2q^{2n})(aq^{n+1}/b; q)_\infty(aq^{n+1}/c; q)_\infty(aq^{n+1}/d; q)_\infty(aq^{n+1}/e; q)_\infty}{(1 - a^2)(abq^n; q)_\infty(acq^n; q)_\infty(adq^n; q)_\infty(aeq^n; q)_\infty}\left(\frac{q}{bcde}\right)^n \tag{1.10}$$
$$= \Pi_q\left[\begin{matrix} a^2q, q, q/a^2, q/bc, q/bd, q/be, q/cd, q/ce, q/de \\ ab, q/ab, ac, q/ac, ad, q/ad, ae, q/ae, q/bcde \end{matrix}\right].$$

2. The functional equation. Denote the left-hand side of (1.10) by $f(b, c, d, e) = f(b, c)$, since we will first consider it as a function of b and c. The part of $f(b, c)$ that depends on b is

$$h_n(b) = \frac{(aq^{n+1}/b; q)_\infty}{(abq^n; q)_\infty b^n}.$$

Observe that

$$h_n(bq) = \frac{(aq^n/b; q)_\infty}{(abq^{n+1}; q)_\infty (bq)^n}$$
$$= h_n(b)\left(1 - \frac{aq^n}{b}\right)\frac{1 - abq^n}{q^n} = h_n(b)\left[q^{-n} - (a/b + ab) + a^2q^n\right],$$

and the only dependence on b in the second factor is in the term independent of n. This can be used to show that

$$f(bq, c) - f(b, cq) = a(c - b)\left(1 - (bc)^{-1}\right)f(b, c). \tag{2.1}$$

Denote the right-hand side of (1.10) by $k(b, c, d, e) = k(b, c)$. It is easy to see that

$$k(bq, c) - k(b, cq) = a(c - b)\left(1 - (bc)^{-1}\right)k(b, c), \tag{2.2}$$

so f and k satisfy the same functional equation. To show they are equal it is necessary to show they have the same initial values.

3. Some special cases. There are a few special cases that can be evaluated using (1.1).

$$(3.1)\quad f(b,\sqrt{q},1,-1) = \sum_{-\infty}^{\infty} \frac{(aq^{n+1}/b;q)_\infty}{(abq^n;q)_\infty}\left(\frac{-q^{1/2}}{b}\right)^n \frac{1}{(1-a^2)}$$

$$= \frac{(aq/b;q)_\infty}{(ab;q)_\infty(1-a^2)}\,{}_1\psi_1\left(\begin{matrix} ab \\ aq/b \end{matrix}; q, -q^{1/2}/b\right)$$

$$= \frac{(-aq^{1/2};q)_\infty(-q^{1/2}/a;q)_\infty(q;q)_\infty(q/b^2;q)_\infty}{(1-a^2)(-q^{1/2}/b;q)_\infty^2(q/ab;q)_\infty(ab;q)_\infty},$$

$$(3.2)\quad f(b,-\sqrt{q},1,-1) = \frac{(aq^{1/2};q)_\infty(q^{1/2}/a;q)_\infty(q;q)_\infty(q/b^2;q)_\infty}{(1-a^2)(q^{1/2}/b;q)_\infty^2(q/ab;q)_\infty(ab;q)_\infty}$$

(3.3)

$$f(b,1,\sqrt{q},-\sqrt{q}) = \frac{2(-a;q)_\infty(-q/a;q)_\infty(q/q)_\infty(q/b^2;q)_\infty}{(1-a^2)(-1/b;q)_\infty(-q/b;q)_\infty(ab;q)_\infty(q/ab;q)_\infty}$$

$$(3.4)\quad f(b,-1,\sqrt{q},-\sqrt{q}) = \frac{2(a;q)_\infty(q/a;q)_\infty(q;q)_\infty(q/b^2;q)_\infty}{(1-a^2)(1/b;q)_\infty(q/b;q)_\infty(ab;q)_\infty(q/ab;q)_\infty}.$$

It is easy, but tedious, to show that the right-hand side of (1.10) reduces to the right-hand side of these identities. For example,

$$k(b,-1,\sqrt{q},-\sqrt{q}) = \frac{(a^2q;q)_\infty(q;q)_\infty(q/a^2;q)_\infty(\sqrt{q}/b;q)_\infty}{(ab;q)_\infty(q/ab;q)_\infty(-a;q)_\infty(-q/a;q)_\infty}$$

$$\cdot\frac{(-\sqrt{q}/b;q)_\infty(-q/b;q)_\infty(-\sqrt{q};q)_\infty(\sqrt{q};q)_\infty(-1;q)_\infty}{(a\sqrt{q};q)_\infty(\sqrt{q}/a;q)_\infty(-a\sqrt{q};q)_\infty(-\sqrt{q}/a;q)_\infty(1/b;q)_\infty}$$

$$= \frac{2(a^2q;q)_\infty(q;q)_\infty(q/a^2;q)_\infty(q/b^2;q^2)_\infty(q^2/b^2;q^2)_\infty(q^2;q^2)_\infty(q;q^2)_\infty}{(ab;q)_\infty(q/ab;q)_\infty(a^2;q^2)_\infty(q^2/a^2;q^2)_\infty(a^2q;q^2)_\infty(q/a^2;q^2)_\infty(1/b;q)_\infty}$$

$$\cdot\frac{(a;q)_\infty(q/a;q)_\infty}{(q;q)_\infty(q/b;q)_\infty}$$

$$= \frac{2(a^2q;q)_\infty(q;q)_\infty(q/a^2;q)_\infty(q/b^2;q)_\infty(q;q)_\infty(a;q)_\infty(q/a;q)_\infty}{(ab;q)_\infty(q/ab;q)_\infty(a^2;q)_\infty(q/a^2;q)_\infty(1/b;q)_\infty(q;q)_\infty(q/b;q)_\infty}$$

$$= \frac{2(a;q)_\infty(q/a;q)_\infty(q;q)_\infty(q/b^2;q)_\infty}{(1-a^2)(1/b;q)_\infty(q/b;q)_\infty(ab;q)_\infty(q/ab;q)_\infty},$$

where

$$(-a;q)_\infty(a;q)_\infty = (a^2;q^2)_\infty \quad\text{and}\quad (a;q^2)_\infty(qa;q^2)_\infty = (a;q)_\infty$$

were used.

4. Completion of the proof. Let $P(j, k, m, n)$ denote the proposition that

$$f\left(q^{1/2-j}, -q^{1/2-k}, q^{-m}, -q^{-n}\right) = k\left(q^{1/2-j}, -q^{1/2-k}, q^{-m}, -q^{-n}\right).$$

We have shown that $P(j,0,0,0)$, $P(0,k,0,0)$, $P(0,0,m,0)$ and $P(0,0,0,n)$ hold when $j, k, m, n = 1, 2, \ldots$. The functional equations (2.1) and (2.2) show that $P(j,1,0,0)$ holds, then that $P(j,2,0,0)$ holds, and then that $P(j,k,0,0)$ holds when $j, k \geqslant 0, j + k > 0$. Then using $P(0,k,0,0)$ and $P(0,0,m,0)$ and (2.1) and (2.2), we have $P(0,k,m,0)$, $k, m \geqslant 0, k + m > 0$. Then $P(j,k,m,0)$ holds when $j, k, m \geqslant 0$, $j + k + m > 0$, and finally $P(j,k,m,n)$ holds when $j, k, m, n \geqslant 0, j + k + m + n > 0$. To conclude that (1.10) holds under the appropriate convergence conditions, we need only remark that both sides are analytic in b^{-1}, c^{-1}, d^{-1} and e^{-1} and agree for infinitely many values.

References

1. G. Andrews, *Applications of basic hypergeometric series*, SIAM Rev. **16** (1974), 441–484.

2. G. Andrews and R. Askey, *Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials*, Higher Combinatorics (M. Aigner, ed.), Reidel, Dordrecht and Boston, Mass., 1977, pp. 3–26.

3. R. Askey, *Ramanujan's extensions of the gamma and beta functions*, Amer. Math. Monthly **87** (1980), 346–359.

4. ______, *An elementary evaluation of a beta type integral*, Indian J. Pure Appl. Math. **14** (1983), 892–895.

5. R. Askey and M. E.-H. Ismail, *The very well poised ${}_6\psi_6$*, Proc. Amer. Math. Soc. **77** (1979), 218–222.

6. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. (to appear).

7. W. N. Bailey, *Series of hypergeometric type which are infinite in both directions*, Quart. J. Math. Oxford Ser. **7** (1936), 105–115.

8. M. E.-H. Ismail, *A simple proof of Ramanujan's ${}_1\psi_1$ sum*, Proc. Amer. Math. Soc. **63** (1977), 185–186.

9. S. Ramanujan, *Notebooks of Srinivasa Ramanujan*, Vol. 2, Tata Institute of Fundamental Research, Bombay, 1957.

10. L. J. Slater and A. Lakin, *Two proofs of the ${}_6\psi_6$ summation theorem*, Proc. Edinburgh Math. Soc. (2) **9** (1953–57), 116–121.

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706