PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 4, April 1984

SYLVESTER'S PROBLEM AND MOTZKIN'S THEOREM FOR COUNTABLE AND COMPACT SETS

PETER B. BORWEIN[1]

ABSTRACT. The following three variations of Sylvester's Problem are established. Let A and B be compact, countable and disjoint sets of points.

(1) If A spans E^2 (the Euclidean plane) then there must exist a line through two points of A that intersects A in only finitely many points.

(2) If A spans E^3 (Euclidean three-space) then there must exist a line through exactly two points of A.

(3) If $A \cup B$ spans E^2 then there must exist a line through at least two points of one of the sets that does not intersect the other set.

Introduction. A question of captivating simplicity that has attracted considerable attention since it was originally posed by J. J. Sylvester in 1893 [**8**] is the following:

SYLVESTER'S PROBLEM. *If A is a finite noncollinear set of points in the plane then there exists a line through exactly two points of A.*

A related question posed by R. Graham and answered by T. Motzkin [7] provides the next result.

MOTZKIN'S THEOREM. *Let A and B be two finite disjoint sets of points in the plane. Then either $A \cup B$ lies on a line or there exists a line through at least two points of one of the sets that does not intersect the other set.*

The large corpus of literature on Sylvester's Problem and many of its generalizations may be accessed through the references in [**4** or **6**]. Likewise, a discussion of extensions of Motzkin's Theorem is to be found in [3] and the references therein. Proofs of both results may be found in [**4**].

This paper will discuss analogues of these results where the sets in question are countable and compact. In answer to a question raised in [**1** and **9**], we show that Motzkin's Theorem generalizes directly; that is, we need only assume that the sets A and B in Motzkin's Theorem are countable, compact and disjoint. (See Proposition 3.)

In contrast, the following example (from the folk literature) shows that the conclusion of Sylvester's Problem no longer holds if we assume A is countable instead of finite. Consider the following set in the real *projective* plane:

$$A = \{(0,k), (1,2m), (-1,2n) | k, m, n \text{ integers}\}.$$

Received by the editors November 10, 1982 and, in revised form, April 29, 1983.
1980 *Mathematics Subject Classification*. Primary 51M05.

[1]Research supported in part by N.S.E.R.C. of Canada.

The addition of a single point "at infinity" makes A compact and thus provides an example of a countable and compact set in the real projective plane for which the conclusion of Sylvester's Problem is false. This example, which can easily be projected to lie in E^2, appears to be the only example of its kind and it would be of interest to know if any others exist.

What we will show is that a noncollinear countable and compact set in the (real) plane E^2 must determine a line through finitely many of its points. We also show that the conclusion of Sylvester's Problem holds for countable and compact sets in Euclidean three-space E^3. (See Propositions 1 and 2.)

For the remainder of the paper we will be working exclusively in Euclidean spaces.

2. Sylvester's Problem generalized. Our first generalization is

PROPOSITION 1. *A noncollinear, compact and countable set A in the plane cannot have the property that every line through two points of A intersects A in infinitely many points.*

PROOF. We proceed by contradiction. Suppose every line through two points of A contains infinitely many points of A. Let $c \in A' - A''$ (where A' and A'' are the first and second derived sets of A, respectively). Since A' is nonempty and at most countable, such a c exists. Let $\{c_i\}$ be a sequence of points in A with limit c. Let l be any line that does not contain c. Consider the projection π from the point c to the line l. Let p be any point of $A - \{c\}$ that does not lie on a line through all but finitely many of the c_i. Consider the lines joining the $\{c_i\}$ and p. (See Figure 1.) Each such line contains infinitely many points of A and hence contains a limit point $q_i \in A'$. The sequence $\{q_i\}$ has a convergent subsequence $\{q_i'\}$. These q_i' tend to a point q on the line joining c and p and, since $c \notin A''$, the q_i' do not tend to c. One now verifies that $\pi(q_i')$ tends to $\pi(p)$. Let N_c be an open neighbourhood of c such that $A - N_c$ contains at least three noncollinear points. By the above argument every point of $\pi(A - N_c)$ is a limit point of $\pi(A - N_c)$ (except perhaps one point, d, in the case that all but finitely many of the c_i are collinear). Since π is continuous and $A - N_c$ is countable and compact, we see that $\pi(A - N_c)$ is a nonempty countable perfect set, which is a contradiction. (In the case where there exists a single point d which is not a limit point of $\pi(A - N_c)$, we have the contradiction that $\pi(A - N_c) - d$ is countable and perfect.) This finishes the proof. □

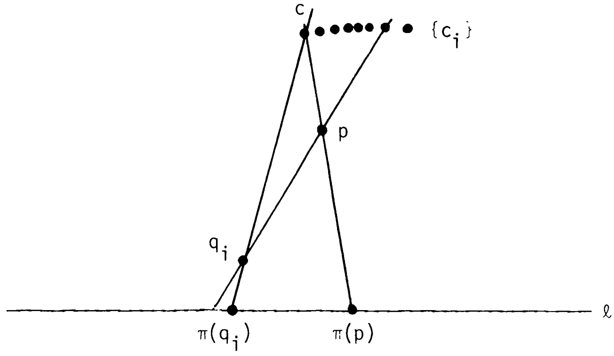

FIGURE 1

As a consequence of this we can prove that the conclusion of Sylvester's Problem is valid for countable and compact sets in three dimensions.

PROPOSITION 2. *A noncoplanar, compact and countable set A in E^3 must contain two points so that the line through these points passes through no other point of A.*

PROOF. Suppose, for the sake of contradiction, that every line through two points of A contains a third point of A. Thus, by the contrapositive of the planar version of Sylvester's Problem mentioned in the introduction, every plane through three noncollinear points of A must contain infinitely many points of A. Consider the projection π from an isolated point $a \in A$ to any plane not containing a. Let b and c be points of A and suppose a, b and c are not collinear. Let P be the plane defined by a, b and c. We must first show that $P \cap A$ does not lie on finitely many lines through a.

Once again we proceed by contradiction. Suppose $P \cap A$ lies on finitely many lines, $l_1, l_2, \ldots, l_n$, all through the isolated point a. Let p be a limit point of the infinite compact set $P \cap A$ and suppose p lies on l_1. Let q be any point in $(P \cap A) - l_1$. Since p is a limit point, there exists a sequence $\{p_i\}$ in $P \cap A$ that tends to p and, by the assumptions, these points can all be assumed to lie on l_1. Consider the projection Γ from p to l_n. If we consider the lines joining the p_i to q, we can see that each of these lines must contain a third point of $(P \cap A) - l_1$ and, hence, that $\Gamma(q)$ is a limit point of $\Gamma(P \cap A)$. This leads to the contradiction that $\Gamma(P \cap A) - a$ is a countable perfect set.

We can now assert that the image of P under the projection π contains infinitely many points. Thus, any line through two points of $\pi(A)$ contains infinitely many points of $\pi(A)$. This contradicts Proposition 1 and finishes the proof. $\square$

Some of these results were announced in [**2**].

3. Motzkin's Theorem generalized.

PROPOSITION 3. *Let A and B be two disjoint, compact, countable subsets of E^2. Then either $A \cup B$ lies on a line or there exists a line through two points of one of the sets that does not intersect the other set.*

PROOF. We assume the proposition is false. Thus, we are assuming $A \cup B$ spans E^2 and every line through two points of either of the sets intersects the other set. We may also, by Motzkin's Theorem, assume $A \cup B$ is infinite. If T is closed and countable then the Cantor-Bendixon Theorem (see [**5**, p. 133]) says there is a countable ordinal γ such that $T^{(\gamma)} = \varnothing$. Note that the derived sets corresponding to the limit ordinals are defined by intersection.

The statement we prove by induction is the following: If p is an element of $A^{(\gamma)}$, q and r are points of A, and p, q and r are not collinear, then either $l(p, q)$ or $l(p, r)$ contains a point of $B^{(\gamma)}$. Here, $l(s, t)$ denotes the line joining s and t.

Suppose λ is not a limit ordinal and $p \in A^{(\lambda)}$. Then there exist $p_i \in A^{(\lambda-1)}$ such that $p_i \to p$. (See Figure 2.) If p_i is sufficiently close to p so that p_i, q and r are not collinear, then, by the inductive hypothesis, one of $l(p_i, q)$ or $l(p_i, r)$ contains a point of $B^{(\lambda-1)}$. From this and the compactness of B, it follows that one of the lines $l(p, q)$ or $l(p, r)$ contains a limit point of points of $B^{(\lambda-1)}$ as is required.

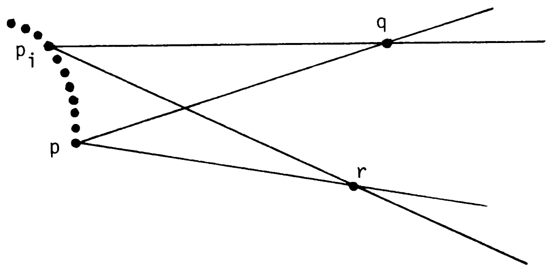

FIGURE 2

When λ is a limit ordinal the argument follows in an analogous fashion. In this case one of the lines $l(p, q)$ or $l(p, r)$ will contain a B limit point of each lower order and, thus, one of the lines contains a B limit point of order λ. This finishes the induction. Note that we may interchange the roles of A and B in this argument.

Before we invoke the Cantor-Bendixon Theorem and finish the proof we must attend to three details that rule out certain degenerate cases.

(1) First, we observe that both sets must span the plane. If A, for example, lies on a line l, then connecting the B point, which must lie on l, to any B point not on this line generates a point of A not on l. This, in conjunction with the inductive step, guarantees that both sets must be infinite.

(2) Next, we show that it is not possible that all but finitely many A points lie on a line and that all but finitely many B points also lie on a line. Suppose all but finitely many A points (O_s) lie on l_1 and all but finitely many B points (X_s) lie on l_2. (See Figure 3.) Since l_1 contains a B point and l_2 contains an A point, one of these points cannot lie at their intersection. So we may suppose there is an A point, a, on l_2 and not on l_1. However, connecting a to the infinitely many points of A on l_1 generates infinitely many B points not on l_2.

(3) Thirdly, we observe, by contradiction again, that it is not possible that A lies on the union of two lines and that B also lies on the union of two lines. To see this, observe that, of the four points of intersection of these pairs of lines, two must be from A and two must be from B. Now the line joining the two A points contains a B point that cannot lie on any of the four lines.

We are now in a position to finish the proof.

The Cantor-Bendixon Theorem tells us there exists a first ordinal δ so that $A^{(\delta)} \cup B^{(\delta)} = \varnothing$. Furthermore, this is not a limit ordinal since the derived sets corresponding to these ordinals are nested intersections of compact sets and hence, nonempty. Let $a \in A^{(\delta-1)} \cup B^{(\delta-1)}$ and assume $a \in A^{(\delta-1)}$. By the first observation above and the inductive hypothesis there must exist at least one point $b \in B^{(\delta-1)}$. From the third observation we deduce that one of $\dot{B}^{(\delta-1)}$ or $A^{(\delta-1)}$ must contain at least two points. In fact, if we consider the pencils of lines originating from a and b we can deduce that both $B^{(\delta-1)}$ and $A^{(\delta-1)}$ contain at least two points (in order to do this it suffices to show that each pencil consists of at least three lines). Now by the second observation there must exist a point of (say) $A^{(\delta-1)}$ and infinitely many A

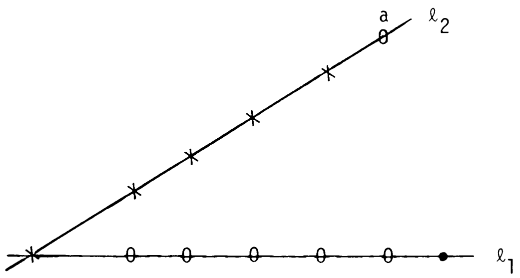

FIGURE 3

points $\{a_i\}$ so that all the lines $l(a, a_i)$ are distinct. Again by the induction we must have, on these lines, infinitely many points $\{b_i\}$, each in $B^{(\delta-1)}$. These $\{b_i\}$ have a limit point in $B^{(\delta)} = \varnothing$. This contradiction finishes the proof. $\square$

REFERENCES

1. J. M. Borwein, *Problem* 297, Canad. Math. Bull. **23** (1980), 506.

2. P. B. Borwein, *Variations on Sylvester's Problem*, Proc. Atlantic Mathematics Days, Memorial University (1981), 39–43.

3. P. B. Borwein and M. Edelstein, *A conjecture related to Sylvester's Problem*, Amer. Math. Monthly **90** (1983), 389–390.

4. G. D. Chakerian, *Sylvester's problem on collinear points and a relative*, Amer. Math. Monthly **77** (1970), 164–167.

5. E. Kamke, *Theory of sets*, Dover, New York, 1950.

6. W. Moser, *Research problems in discrete geometry*, McGill Math. Rep. 81-3, McGill Univ., Montreal, Quebec.

7. T. S. Motzkin, *Nonmixed connecting lines*, Notices Amer. Math. Soc. **14** (1967), 837.

8. J. J. Sylvester, *Mathematical question* 11851, Educational Times **59** (1893), 98.

9. D. Tingley, *Topics related to Sylvester's Problem*, M. A. Thesis, Dalhousie Univ., Halifax, N.S., 1976.

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTING SCIENCE, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA B3H 4H8