A FILTER ON $[\lambda]^*$

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Abstract. We define a filter on $[\lambda]^*$ with properties similar to those of the closed unbounded filter in $P_\kappa(\lambda)$. This filter’s behaviour depends on set theoretical hypotheses.

The study of the combinatorial properties of the collection of subsets of uncountable cardinals has been a main line of research in the theory of large cardinals. For $\kappa < \lambda$ regular uncountable cardinals, the space $P_\kappa(\lambda)$ is the collection of subsets of $\lambda$ of cardinality smaller than $\kappa$. This space was introduced in the investigation of strongly compact cardinals and of supercompact cardinals. In [Jel] Jech studied the space $P_\kappa(\lambda)$ on its own and obtained interesting generalizations to the context of this space of classical results pertaining to the theory of the space $\kappa$.

The space $[\lambda]^*$, the collection of subsets of $\lambda$ of cardinality $\kappa$, arises in the investigation of the so-called huge cardinals. As shown in Solovay, Reinhardt and Kanamori [SRK], $\kappa$ is huge with target $\lambda$ if and only if there exists a $\kappa$-complete normal fine ultrafilter on $[\lambda]^*$. We recall the definition of huge cardinal. We say that $\kappa$ is huge with target $\lambda$ if there is an elementary embedding $j: V \to M$ of the universe into a transitive model $M$ containing all the ordinals such that $\kappa$ is the critical point of $j$, $j(\kappa) = \lambda$ and $\lambda^M \subseteq M$. We denote this by $\kappa \rightarrow (\lambda)$. (See [BDPT].) In this case the axiom of choice allows us to show that the set $(\lambda)^* = \{P \subseteq \lambda | \text{order type of } P = \kappa\}$ belongs to the normal ultrafilter on $[\lambda]^*$, and thus we can characterize the fact that $\kappa \rightarrow (\lambda)$ by the fact that there exists a normal, $\kappa$-complete, fine ultrafilter on $(\lambda)^*$. Thus, under the axiom of choice the first characterization implies the second. This is not so in the absence of the axiom of choice; for instance, under the axiom of determinateness the implication fails, as shown by Mignone in [Mig].

A natural problem is to find a filter on $[\lambda]^*$ analogous to the closed unbounded filter for $P_\kappa(\lambda)$ constructed by Jech in [Jel]. The filter we construct is a $\kappa$-complete, normal, fine, nontrivial filter, and, as shown by J. Baumgartner, it is the smallest filter on $[\lambda]^*$ with these properties. Under the assumption that $\kappa$ is huge with target $\lambda$, all elements of our filter have measure 1 with respect to the normal measure.

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generated by the witnessing elementary embedding. Thus our filter exhibits a behaviour similar to the closed unbounded filter in $P_\kappa(\lambda)$.

Since the space $[\lambda]^\kappa$ is simply $P_\kappa(\lambda) - P_\kappa(\lambda)$, one could ask why we consider a new notion of a closed unbounded set at all since we have Jech's notion of a closed unbounded set in $P_\kappa(\lambda)$ and could restrict it to $[\lambda]^\kappa$. The reason for seeking another filter is that under $\kappa \rightarrow (\lambda)$ not all the closed unbounded sets in the sense of Jech have measure 1, and, in fact, the cones over sets of cardinality $\kappa$ have measure 0. Thus a different notion is necessary.

The behaviour of our filter is not a simple one. As we said, if $\kappa \rightarrow (\lambda)$ then our filter is included in every fine normal measure on $[\lambda]^\kappa$, but if the universe is not too fat (for instance if $V = L$) our filter is just the closed unbounded filter on $P_\kappa(\lambda)$.

We acknowledge the helpful remarks made by J. Baumgartner. He proved Theorem 1(e) using a property of closed unbounded sets of $P_\kappa(\lambda)$ which uses functions from $[\lambda]^{<\omega}$ into $\lambda$ and which can be obtained from Menas' basis result for closed unbounded sets [Me1]. We follow, in general, the notation of Jech in [Je2]. If $a \in P_\kappa(\lambda)$, $\hat{a} = \{ p \in P_\kappa(\lambda) | a \subseteq p \}$ and $\hat{a} = \{ P \in [\lambda]^\kappa | a \subseteq P \}$.

1. The filter $\mathcal{F}_{\kappa,\lambda}$. Given a set $X \subseteq P_\kappa(\lambda)$, define $A_X$, the basic set generated by $X$, as follows:

\[ (*) \quad A_X = \{ P \in [\lambda]^\kappa | \text{there exists a directed system } D \subseteq X \text{ such that } P = \bigcup D \}. \]

We now define the filter $\mathcal{F}_{\kappa,\lambda}$ as follows: $A \in \mathcal{F}_{\kappa,\lambda}$ if and only if there is a closed and unbounded subset $X$ of $P_\kappa(\lambda)$ such that $A_X \subseteq A$.

**Theorem 1.** The filter $\mathcal{F}_{\kappa,\lambda}$ possesses the following properties:

(a) The cones $\hat{a}$ (for $a \in P_\kappa(\lambda)$) belong to $\mathcal{F}_{\kappa,\lambda}$.

(b) $\mathcal{F}_{\kappa,\lambda}$ is $\kappa$-complete.

(c) $\mathcal{F}_{\kappa,\lambda}$ is normal; Fodor's property holds for $\mathcal{F}_{\kappa,\lambda}$-stationary sets.

(d) If $\kappa \rightarrow (\lambda)$ and $\mu$ is the normal measure induced on $[\lambda]^\kappa$ by a witnessing embedding, then every set in $\mathcal{F}_{\kappa,\lambda}$ has $\mu$-measure 1. In this case $\mathcal{F}_{\kappa,\lambda}$ is not $\kappa^+$-complete.

(e) $\mathcal{F}_{\kappa,\lambda}$ is the least $\kappa$-complete, normal, fine filter on $[\lambda]^\kappa$.

**Proof.** (a) If $a \in P_\kappa(\lambda)$ the the cone $\hat{a}$ over $a$ in $[\lambda]^\kappa$ is exactly $A_\hat{a}$.

(b) Obvious.

(c) Let $(A_\alpha | \alpha < \lambda)$ be a $\lambda$-sequence of elements of $\mathcal{F}_{\kappa,\lambda}$. Choose, for each $\alpha < \lambda$, a set $X_\alpha$ closed and unbounded in $P_\kappa(\lambda)$ such that $A_{X_\alpha} \subseteq A_\alpha$. Define now $Y = \Delta_{\alpha < \lambda} X_\alpha$. We show that $A_Y \subseteq \Delta_{\alpha < \lambda} A_{X_\alpha}$. It is enough to verify that $A_Y \subseteq \Delta_{\alpha < \lambda} A_{X_\alpha}$. The latter set is $\{ P \in [\lambda]^\kappa |$ for all $\xi \in \alpha$, $P \in A_{X_\alpha} \} = \{ P \in [\lambda]^\kappa |$ for all $\xi \in \alpha$ there is a directed system $D_\xi \subseteq X_\alpha$ such that $P = \bigcup D_\xi \}. \}$ Let $P \in A_Y$, and pick $D \subseteq Y$. Given $\xi \in P$, put $D_\xi = \{ a \in P_\kappa(\lambda) | a \in D \text{ and } \xi \in a \}$. Then $D_\xi$ is directed for each $\xi \in P$, since $D \subseteq \Delta_{\alpha < \lambda} X_\alpha = Y$, $D_\xi \subseteq X_\alpha$, and, finally, $\bigcup D_\xi = \bigcup D$.

(d) Assume now that $\kappa \rightarrow (\lambda)$. Let $\mu$ be the normal measure in $[\lambda]^\kappa$ generated by the witnessing embedding $j$; $\mu$ is defined by $A \in \mu \Rightarrow j^* \lambda \in j(A)$. Thus we must show that $j^* \lambda \in j(A_X)$ whenever $X \subseteq P_\kappa(\lambda)$ is a closed unbounded set. By the elementarity of $j$ we have that if $j: V \rightarrow M$, then $j(A_X) = A_{j(X)}^M$. We need
to show that $M \not\models j'' \lambda \in A_{j(\lambda)}$, i.e. we need to exhibit in $M$ a directed system contained in $A_{j(\lambda)}$ whose union is $j'' \lambda$. This system is simply $\{ j(a) \mid a \in X \}$.

Under the hypothesis that $\kappa \rightarrow (\lambda)$, the filter $\mathcal{F}_{\kappa, \lambda}$ is not $\kappa^+$-complete. In fact, put $A_\xi = \{ \xi, \kappa + \xi \}$ (the cone in $[\lambda]^\kappa$ over $\{ \xi, \kappa + \xi \}$). If $\mathcal{F}_{\kappa, \lambda}$ were $\kappa^+$-complete then we would have that $\bigcap_{\xi < \kappa} A_\xi = \kappa + \kappa$ is in the filter and, therefore, that it has measure 1. This is absurd since $(\lambda)^\kappa \cap \kappa + \kappa = \emptyset$. From the proof it follows that $A_X \in \mu$ for every $X$ unbounded in $P_\kappa(\lambda)$.

(Note that if $X$ is unbounded then $A_X$ is $\mathcal{F}_{\kappa, \lambda}$ stationary. This follows from the fact that a hand-over-hand construction with alternating choices from a given closed and unbounded set and $X$ is possible. In fact, the family of $A_X$'s for $X$ unbounded generates a $\kappa$-complete filter. In opposition to Theorem 1(c), this filter does not seem to be normal.)

To prove (e) we need to establish some facts: Following Menas [Mel], given a function $f: [\lambda]^n \rightarrow P_\kappa(\lambda)$ let $C(f) = \{ p \in P_\kappa(\lambda) \mid f(x_1, \ldots, x_n) \subseteq p \text{ for all } x_1, \ldots, x_n \text{ contained in } p \}$. For any $f: [\lambda]^n \rightarrow P_\kappa(\lambda)$, $C(f)$ is a closed and unbounded subset of $P_\kappa(\lambda)$.

Also, let $C(f) = \{ p \in [\lambda]^n \mid f(x_1, \ldots, x_n) \subseteq p \text{ for } x_1, \ldots, x_n \text{ in } p \}$. Given $f: [\lambda]^n \rightarrow P_\kappa(\lambda)$, $A_{C(f)} = C(f)$. To prove this, let $P \in A_{C(f)}$. There is a directed set $D \subseteq C(f)$ such that $P = \bigcup D$. Now, if $\{ x_1, \ldots, x_n \} \subseteq P$, there is $p \in D$ such that $\{ x_1, \ldots, x_n \} \subseteq p$, but then $f(x_1, \ldots, x_n) \subseteq p \subseteq P$. Conversely, given $P \in C(f)$, it is easy to show that $C(f) \cap P(P)$ is a directed subset of $C(f)$ with union $P$.

Menas showed in [Mel] that for any $X \subseteq P_\kappa(\lambda)$ closed and unbounded there is an $f: [\lambda]^2 \rightarrow P_\kappa(\lambda)$ such that $C(f) \subseteq X$. From this follows

**Lemma 2 (Basis property).** A set $A$ belongs to the filter $\mathcal{F}_{\kappa, \lambda}$ if and only if there is a function $f: [\lambda]^2 \rightarrow P_\kappa(\lambda)$ such that $C(f) \subseteq A$.

Now (e) follows. Indeed, if $\mathcal{F}_{\kappa, \lambda}$ is not the least $\kappa$-complete, normal, fine filter on $[\lambda]^\kappa$ then let $\mathcal{F} \subseteq \mathcal{F}_{\kappa, \lambda}$ be the least such filter. Let $A \in \mathcal{F}_{\kappa, \lambda} - \mathcal{F}$. Clearly both $A$ and $[\lambda]^\kappa - A$ are $\mathcal{F}$-stationary (both meet each element of $\mathcal{F}$). By Lemma 2 there is $f: [\lambda]^2 \rightarrow P_\kappa(\lambda)$ such that $C(f) \subseteq A$. Thus $([\lambda]^\kappa - A) \cap C(f) = \emptyset$. Applying normality of $\mathcal{F}$ twice, we find an $\mathcal{F}$-stationary set $B \subseteq ([\lambda]^\kappa - A)$ and a fixed pair of ordinals $x_1, x_2$ in $\lambda$ such that for every $P \in B$, $\{ x_1, x_2 \} \subseteq P$ and $f(x_1, x_2) \subseteq P$. But then, $B \cap f(x_1, x_2) = \emptyset$, contradicting the fact that $\mathcal{F}$ is fine. □

We remark that the filter $\mathcal{F}_{\kappa, \lambda}$ is never $\kappa^{++}$ complete (for each $\alpha < \kappa^+$ take $\{ \alpha \} = \{ P \in [\lambda]^\kappa \mid \alpha \in P \}$; all these sets are in $\mathcal{F}_{\kappa, \lambda}$ but $\bigcap_{\alpha < \kappa} \{ \alpha \} = \emptyset$).

As in the case of $P_\kappa(\lambda)$ (see Jech [Je1]) we tailored our definition of closed unbounded sets of $[\lambda]^\kappa$ to be able to show that all elements of $\mathcal{F}_{\kappa, \lambda}$ have measure 1 under normal measures on $[\lambda]^\kappa$. It is known that the definition of closed unbounded subset of $P_\kappa(\lambda)$ can be weakened by showing that an unbounded set is closed under directed systems if and only if it is closed under unions of chains (see Magidor [Ma]). The same phenomenon occurs for $[\lambda]^\kappa$ since we have the following closure lemma.
Lemma 3. Given \( \kappa \) an uncountable regular cardinal, let \( \{a_\xi| \xi < \kappa\} \) be a sequence of length \( \kappa \) such that \( \{a_\xi| \xi < \kappa\} \) is a directed subset of \( P_\kappa(\lambda) \). Then there exist arbitrarily large initial segments \( \{a_\xi| \xi < \eta\} (\eta < \kappa) \) which are directed.

Given a directed \( D \subseteq P_\kappa(\lambda) \), if \( |\bigcup D| = \kappa \) and \( |D| > \kappa \) it is easy to construct \( D' \subseteq D \) with \( |D'| = \kappa \) and \( \bigcup D' = \bigcup D \). Thus we may always assume our directed systems have cardinality \( \kappa \). We use Lemma 3 to split \( D \) into an increasing chain of directed subsystems each of size smaller than \( \kappa \). The union of each of these subsystems is in \( X \). Thus, given \( P \in A_X \), we can present \( P \) as the union of an increasing \( \kappa \)-chain of elements of \( X \). We thus have

Theorem 4. If statement \((*)\) of the definition of \( \mathcal{G}_{\kappa,\lambda}^X \) is replaced by
\[(*)' \quad A'_X = \{P \in [\lambda]^\kappa| P \text{ is the union of an increasing } \kappa \text{-chain of elements of } X\},\]
we obtain the same filter.

The argument of the proof does not go through if \( X \) is only unbounded and not closed. In fact, if \( \kappa \rightarrow (\lambda) \) then the following set \( E \) is unbounded but \( \mathcal{A}'_\kappa \) is not in the filter (it has normal measure zero). Let
\[E = \{p| \exists \xi \forall \xi < \lambda \text{ if } p \cap [\kappa \cdot \xi, \kappa \cdot (\xi + 1)] \neq \emptyset, \text{ then } p \cap [\kappa \cdot \xi, \kappa \cdot (\xi + 1)] = (\kappa \cdot \xi) + \alpha - (\kappa \cdot \xi) \text{ and } \{\xi|p \cap [\kappa \cdot \xi, \kappa \cdot (\xi + 1)] \neq \emptyset\} = \kappa \cap p = \alpha\}.

Clearly if \( P \in \mathcal{A}'_\kappa \) then o.t. \( P > \kappa \). So \( \mathcal{A}'_\kappa \cap (\lambda)^\kappa = \emptyset \) [DPM].

Assume \( \kappa \) is huge with a target \( \lambda \) and \( \nu \) a corresponding normal measure in \( [\lambda]^\kappa \) and that, in addition, there exists a normal measure \( \mu \) in \( P_\kappa(\lambda) \) with the partition property (this happens, for instance, if \( \kappa \) is twice huge with \( \lambda \) a first target). The set \( E \) has \( \mu \)-measure 0. However since the measure \( \mu \) has the partition property there exists a set \( X \), of \( \mu \)-measure 1 such that \( p, q \in X \) and \( p \subseteq q \Rightarrow |p| < |q \cap \kappa| \text{ (cf. [Me2]).} \)
The set \( A'_X \) is not in our filter (otherwise \( A'_X \cap (\lambda)^\kappa \) has \( \nu \)-measure 1, which is absurd).

We will now prove a lemma which implies that, just as in the case of the closed unbounded filter in \( P_\kappa(\lambda) \), it is enough to apply twice the operation \( \Delta \) (diagonal intersection) to cones to obtain all sets of the form \( A_X \) for \( X \) closed and unbounded in \( P_\kappa(\lambda) \).

Lemma 5. Given a collection \( \{X_\xi| \xi < \lambda\} \) of closed unbounded sets in \( P_\kappa(\lambda) \),
\[\Delta_{\xi < \lambda} A_{X_\xi} = A_{\Delta_{\xi < \lambda} X_\xi}.
\]

Proof. It is enough to prove \( \Delta_{\xi < \lambda} A_{X_\xi} \subseteq A_{\Delta_{\xi < \lambda} X_\xi} \). We first prove the following:

Fact 6. If \( P \in \Delta_{\xi < \lambda} A_{X_\xi} \), then for every \( q \in P_\kappa(P) \), the intersection \( \bigcap_{\xi \in q} X_\xi \) is closed and unbounded in \( P_\kappa(P) \).

Indeed, given \( p \in P_\kappa(P) \), we perform an induction of length \( |q| \cdot \omega \) to cover \( p \) with elements of each \( X_\xi (\xi \in q) \). Since \( P \in \Delta_{\xi < \lambda} A_{X_\xi} \), for every \( \alpha \in P \) there is an increasing \( \kappa \)-chain of elements of \( X_\alpha \) with union \( P \). Therefore if \( q = \{\alpha_\nu| \nu < |q|\} \), let \( p_0^0 = p \), and for each \( \nu < |q| \) let \( p_\nu^0 \in X_{\alpha_\nu} \) be such that \( \bigcup_{\xi < \nu} p_\xi^0 \subseteq p_\nu^0 \subseteq P \). Similarly,
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put $p_0^{\kappa+1} = \bigcup_{\xi<\kappa} p_\xi^\kappa$ and $p_\xi^{\kappa+1} \in X_{\alpha_\xi}$ such that $\bigcup_{\xi<\kappa} p_\xi^{\kappa+1} \subseteq p_{\kappa+1}^\kappa \subseteq P$. The sets $p_\xi^\kappa$ can always be found as $P$ is the limit of $\kappa$-chains of elements of each $X_{\alpha_\xi}$. Finally, $\bigcup_{\eta<\kappa} r \in [\lambda]^\kappa p_\eta^\kappa$ is a subset of $P$ which contains $p$ and belongs to $\cap_{\xi<\kappa} X_\xi$. This completes the proof of the Fact.

To complete the proof of Lemma 5 it is enough to show that if $P \in \Delta_{\xi<\lambda} A_{X_\xi}$ and $p \in P_\ell(P)$ then there is a $q \in \Delta_{\xi<\lambda} X_\xi$ such that $p \subseteq q \subseteq P$. Using Fact 6 we find such $q$ as follows: Let $q_0 = p$, $q_{n+1}$ is a set in $\bigcap_{\xi<\alpha_\xi} X_\xi$ such that $q_n \subseteq q_{n+1} \subseteq P$. Let $q = \bigcup_{n<\omega} q_n$.

If $C$ is a collection of subsets (of $P_\kappa(\lambda)$ or $[\lambda]^\kappa$) we denote by $\Delta C$ the collection consisting of diagonal intersections of elements of $C$.

Corollary 7. $[\lambda]^\kappa: X$ is closed and unbounded in $P_\kappa(\lambda)) = \Delta \Delta \{p | p \in P_\kappa(\lambda))\}.

Proof. D. Carr [Ca] showed that closed unbounded subsets of $P_\kappa(\lambda)$ are just elements of $\Delta \Delta \{p | p \in P_\kappa(\lambda))\}$. So given $X \subseteq P_\kappa(\lambda)$ closed and unbounded, $X = \Delta_{\xi<\lambda} \Delta_{\eta<\lambda} \tilde{p}_{\xi,\eta}$. Now we apply Lemma 5 twice. □

Consider now the following operator $A'_\kappa$:

$$A'_\kappa = \{p \in [\lambda]^\kappa | \text{there is an end extension chain of length } \kappa$$

of elements of $X$, $\langle p_\xi \rangle_{\xi<\kappa}$, such that $P = \bigcup_{\xi<\kappa} p_\xi$$

Clearly the elements of $A'_\kappa$ are of order type $\kappa$ only, and, in fact, a form of the converse property holds as well:

If $\langle p_\xi \rangle_{\xi<\kappa}$ is a $\subseteq$-chain such that $\bigcup_{\xi<\kappa} p_\xi = \kappa$, then there exists a cofinal end extension subchain $\langle p_\eta \rangle_{\eta<\kappa}$.

It follows that if $\kappa \to (\lambda)$ then the filter generated for $A'_\kappa$ (for $X$ closed and unbounded) is exactly $\mathcal{G}_{\kappa,\lambda}$. However, for the set $X$ considered above, $A'_\kappa = \emptyset$ (even though $\nu(A_X) = 1$).

2. Another identity crisis. As we showed in §1, if $\kappa \to (\lambda)$ then $\mathcal{G}_{\kappa,\lambda}$ is included in any fine normal measure on $[\lambda]^\kappa$. In particular, $(\lambda)^\kappa$ is $\mathcal{G}_{\kappa,\lambda}$-stationary since it has measure 1 under any such measure. As we show below this is rather exceptional. First, notice the following:

Proposition 1. $(\lambda)^\kappa$ does not belong to the filter $\mathcal{G}_{\kappa,\lambda}$.

Proof. Construct, for any closed unbounded subset $X$ of $P_\kappa(\lambda)$, an element of $A_X$ of order type greater than $\kappa$. (In fact, for any $\alpha, \kappa \leq \alpha < \kappa^+$, there is an element of $A_X$ of order type greater than $\alpha$.) □

We now prove some auxiliary facts regarding the width of the universe.

Proposition 2. If $0^\#$ does not exist then for every cardinal $\kappa$ and $\lambda \geq \kappa$, whenever $\langle A, \epsilon \rangle <_1 \langle L_\lambda, \epsilon \rangle$, $|A| \geq \kappa$, then $L_\kappa \subseteq A$.

Proof. Otherwise there is $\nu < \kappa$ such that $\nu \not\in A$. Consider now $\eta$ such that $\pi: A \to L_\eta$ is the transitive Mostowski contraction. Then $\eta \geq \kappa$ so $\nu^{+,\eta}$, the successor of
\( \nu \) in the sense of \( L \), is less than or equal to \( \eta \). In particular, \( P(\nu) \cap L \subseteq L_\eta \) so \( \{ \nu \subseteq \nu \mid \nu \in \pi^{-1}(Y) \} \) is an \( L \)-ultrafilter, contradicting the fact that \( 0^* \) does not exist. \( \square \)

Similarly, we prove the following

**Proposition 3.** Let \( \kappa \) be a cardinal and \( a \subseteq \nu \leq \kappa \) such that \( a^* \) does not exist. Then, if \( \langle A, \varepsilon \rangle \prec (L_\lambda[a], \varepsilon) \) and \( \{ a \} \cup \nu \subseteq A \), it follows that \( |A| \geq \kappa \) implies \( L_\lambda[a] \subseteq A \).

(This result can be extended even further, for instance, if \( 0^1 \) does not exist, a similar lemma can be obtained for \( \nu \) greater than the least ordinal measurable in an inner model.)

**Lemma 4.** Let \( \kappa \) and \( a \) be as in Proposition 3 and \( \lambda > \kappa \). Then the cone \( \kappa \) over \( \kappa \) in \( \text{[\lambda]^*} \) belongs to \( \text{[\lambda]}_{\kappa, \lambda} \).

**Proof.** For every \( \xi \geq \kappa \), \( |L_\lambda[a]| = |\xi| \). Enumerate \( L_\lambda[a] \) in order type \( \lambda \) in such a way that \( L_\lambda[a] \) is enumerated by \( \kappa \). Then every subset of \( \lambda \) codes a subset of \( L_\lambda[a] \).

Consider now \( X = \{ p \subseteq \lambda \mid |p| < \kappa \) and \( p \) codes an elementary substructure of \( L_\lambda[a] \) containing \( \{ a \} \cup \nu \} \).

The set \( X \) is clearly closed and unbounded in \( P_\kappa(\lambda) \). If \( P \in A_X \) then \( P \) also codes an elementary substructure of \( L_\lambda[a] \) only now \( |P| = \kappa \). By Proposition 3, the model coded by \( P \) contains \( L_\lambda[a] \), so \( P \) must contain all of \( \kappa \) (since \( \kappa \) enumerates \( L_\lambda[a] \)). We have thus proved that if \( P \in A_X \) then \( \kappa \subseteq P \), that is to say, \( P \in \kappa \). Therefore \( \kappa \in \text{[\lambda]}_{\kappa, \lambda} \). \( \square \)

**Theorem 5.** Under the same assumptions, \( \tilde{P} \in \text{[\lambda]}_{\kappa, \lambda} \) for all \( P \in \text{[\lambda]^*} \), therefore \( (\lambda)^* \) is not \( \text{[\lambda]}_{\kappa, \lambda} \)-stationary.

**Proof.** If \( \pi \) is a permutation of \( \lambda \), it can be extended to \( P_\pi(\lambda) \) and to \( [\lambda]^* \) in the obvious way: if \( a \in P_\pi(\lambda) \), \( \pi(a) = \pi''(a) \), and for \( P \in [\lambda]^* \), \( \pi(P) = \pi''P \). It is clear that \( A_{\pi[X]} = \pi[A_X] \) for a closed unbounded \( X \subseteq P_\pi(\lambda) \). The filter generated by the closed unbounded subsets of \( P_\pi(\lambda) \) and the filter \( \text{[\lambda]}_{\kappa, \lambda} \) are invariant under any permutation of \( \lambda \) and thus, if for any \( P \in [\lambda]^* \), \( \tilde{P} \in \text{[\lambda]}_{\kappa, \lambda} \), then for every \( P \in [\lambda]^* \) we have \( \tilde{P} \in \text{[\lambda]}_{\kappa, \lambda} \). The rest follows easily. \( \square \)

Using this theorem we can proceed toward a connection between \( \text{[\lambda]}_{\kappa, \lambda} \) and the closed unbounded filter on \( P_\kappa(\lambda) \).

**Lemma 6.** If \( X \subseteq P_\kappa(\lambda) \) is closed and unbounded then \( A_X \) is closed under unions of increasing chains of length \( \leq \kappa \).

**Proof.** Let \( \{ P_\xi \}_{\xi < \eta} \) be an \( \eta \)-chain of elements of \( A_X \) with \( \eta < \kappa \), and let \( P = \bigcup_{\xi < \eta} P_\xi \). For each \( \xi < \eta \), \( P_\xi = \bigcup_{\xi' < \xi} P_{\xi'} \), where \( P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{\xi} \subseteq \cdots \), \( \xi < \kappa \), is an increasing \( \kappa \)-chain of elements of \( X \).

We will now construct a \( \kappa \)-chain \( \{ q_\xi \}_{\xi < \kappa} \subseteq X \) such that \( \bigcup_{\xi < \kappa} q_\xi = P \). Each \( q_\xi \) (\( \xi < \kappa \)) will in turn be the union of an \( \eta \)-chain of elements of \( X \),

\[ q_\xi = \bigcup_{\xi < \eta} q_\xi. \]
Define the \( q_{\xi, \delta} \)'s as follows:

\[
q_{0,0} = p_0^0, \quad q_{0, \xi} = \text{the first element of } \left\{ p_{\xi}^{\delta} \right\}_{\delta < \xi} \text{ containing } \bigcup_{\delta < \xi} q_{0, \delta}.
\]

Note that \( q_{0, \xi} \) is defined because \( \bigcup_{\delta < \xi} q_{0, \delta} \) has cardinality \( < \kappa \) and it is contained in \( P_{\xi} \). Since \( P_{\xi} \) has cardinality \( \kappa \) and \( P_{\xi} = \bigcup_{\delta < \xi} p_{\xi}^{\delta} \) there must be a \( p_{\xi}^{\delta} \) containing \( \bigcup_{\delta < \xi} q_{0, \delta} \).

Now put \( q_0 = \bigcup_{\xi < \eta} q_{0, \xi} \). If we have already defined \( q_{\delta} \) for all \( \delta < \xi \) then we define \( q_{\xi, 0} \) as the first element of \( \left\{ p_{\xi}^{\delta} \right\}_{\delta < \kappa} \) covering properly \( \bigcup_{\delta < \xi} q_{0, \delta} \) and \( q_{\xi, \xi} \) as the first element of \( \left\{ p_{\xi}^{\delta} \right\}_{\delta < \kappa} \) covering \( \bigcup_{\tau < \xi} \bigcup_{\delta < \xi} q_{\delta, \tau} \). Now put \( q_{\xi} = \bigcup_{\xi < \eta} q_{\xi, \xi} \).

Clearly, each \( q_{\xi} \in X \), and \( \bigcup q_{\xi} = P \).

For sequences of length \( \kappa \) the construction is somewhat different: Let \( \left\{ p_{\xi}^{\delta} \right\}_{\xi < \kappa} \) be an increasing \( \kappa \)-chain of elements of \( A_X \). As before for each \( \xi < \kappa \), \( p_{\xi} = \bigcup_{\xi < \kappa} p_{\xi}^{\delta} \), where \( \left\{ p_{\xi}^{\delta} \right\}_{\xi < \kappa} \) is an increasing \( \kappa \)-chain from \( X \). We now define a \( \kappa \)-chain \( \left\{ q_{\xi}^{\delta} \right\}_{\xi < \kappa} \subseteq X \) such that \( \bigcup_{\xi < \kappa} q_{\xi}^{\delta} = \bigcup_{\xi < \kappa} p_{\xi} \). Let \( q_0 = p_0^0 \), and if we have defined \( q_{\delta} \) for all \( \delta < \gamma \), put \( q_{\gamma} \) as the first element of \( \left\{ p_{\xi}^{\delta} \right\}_{\xi < \kappa} \) covering \( \bigcup_{\delta < \gamma} q_{\delta} \cup \bigcup_{\delta < \gamma} p_{\delta}^{\alpha} \). Clearly \( \bigcup_{\xi < \kappa} q_{\xi} = \bigcup_{\xi < \kappa} p_{\xi} \) and each \( q_{\xi} \in X \).

**Theorem 7.** Under the hypothesis of Proposition 3, \( \mathcal{F}_{\kappa, \lambda} = \text{CLUB}_{\kappa^+, \lambda} \upharpoonright [\lambda]^\kappa \) (where \( \text{CLUB}_{\kappa^+, \lambda} \) is the filter generated by the closed unbounded subsets of \( P_{\kappa^+}(\lambda) \)).

**Proof.** The previous lemma implies that \( \mathcal{F}_{\kappa, \lambda} \subseteq \text{CLUB}_{\kappa^+, \lambda} \) because under our assumptions all the cones \( \hat{P} \), for \( P \in P_{\kappa^+}(\lambda) \), belong to \( \mathcal{F}_{\kappa, \lambda} \) and therefore each \( A \in \mathcal{F}_{\kappa, \lambda} \) is unbounded in \( P_{\kappa^+}(\lambda) \).

For the other inclusion we use Carr’s result, namely that if \( A \in \text{CLUB}_{\kappa^+, \lambda} \) then \( A = \Delta_{\xi < \lambda} \Delta_{\eta < \lambda} r_{\xi, \eta} \) with \( r_{\xi, \eta} \in P_{\kappa^+}(\lambda) \) (and \( r_{\xi, \eta} \) is its cone in \( P_{\kappa^+}(\lambda) \)). Now, if \( r_{\xi, \eta} \in [\lambda]^\kappa \) then the cone over \( r_{\xi, \eta} \) in \( P_{\kappa^+}(\lambda) \) is just \( r_{\xi, \eta} \), which under our assumption is in \( \mathcal{F}_{\kappa, \lambda} \). It is easy to verify that \( A = \Delta_{\xi < \lambda} \Delta_{\eta < \lambda} r_{\xi, \eta} \), and therefore \( A \in \mathcal{F}_{\kappa, \lambda} \) since \( \mathcal{F}_{\kappa, \lambda} \) is normal.

One way of interpreting these results is that under assumptions like \( V = L \), besides Menas’ basis for the closed unbounded sets in \( P_{\kappa^+}(\lambda) \) there are two other bases. One is sets of the the form \( A_X \) for \( X \) closed and unbounded in \( P_{\kappa}(\lambda) \). Another is sets of the form \( C_{\kappa}(f) \) for \( f: [\lambda]^2 \rightarrow P_{\kappa}(\lambda) \).

These results show that \( \mathcal{F}_{\kappa, \lambda} \) might or might not be the closed unbounded filter on \( P_{\kappa^+}(\lambda) \) restricted to \( [\lambda]^\kappa \) depending on the width of the universe.

**References**


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