PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 4, April 1984

SOME FUNCTION SPACES OF CW TYPE

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ABSTRACT. J. Milnor's result on the CW type of certain function spaces map(X, Y) is extended to allow the case in which X has a finite k-skeleton and $\pi_i Y = 0$, $i > k$. One conclusion is that the self-equivalence monoid of any Postnikov stage of a finite complex has CW type. Another is that the monoid of pointed self-equivalences of a $K(\pi, 1)$ manifold has contractible components when π is finitely-generated.

1. Introduction and statement of results. We shall work entirely in the category of compactly-generated Hausdorff spaces (k-spaces), as described in [**14**], unless explicit exception is noted. Thus, for example, the set map(X, Y) of all maps $X \to Y$ will be topologized by first endowing it with the usual compact-open toplogy and then replacing this with the corresponding k-topology. For another example, a (not necessarily surjective) map of k-spaces will be called a fibration if it satisfies the homotopy-lifting property for all k-spaces.

Let $\underline{W}$ denote the class of spaces having the homotopy type of a CW complex (see [7] for basic facts about CW complexes). The purpose of this note is to prove the following:

1.1. THEOREM. *Let X and Y be connected spaces in $\underline{W}$ and n a nonnegative integer such that*:

(a) *X is homotopy-equivalent to a CW complex with finite n-skeleton, and*

(b) *$\pi_i Y = 0$, for $i > n$.*

Then map(X, Y) *belongs to $\underline{W}$.*

We say that a space satisfying condition 1.1(a) has n-finite type. Path-connected spaces satisfying 1.1(b) will be called $(n + 1)$-co-connected. In [**16, 17**], C. T. C. Wall gives an algebraic characterization of spaces in $\underline{W}$ that have n-finite type. Examples of spaces in $\underline{W}$ having n-finite type, for all n, include: finite complexes, finitely-dominated complexes, nilpotent spaces with finitely-generated homology groups, and connected spaces with finite fundamental group and finitely-generated higher homotopy groups.

Received by the editors December 16, 1982.
1980 *Mathematics Subject Classification*. Primary 55P99.
Key words and phrases. Homotopy type, CW complex, function space.
[1]Partly supported by NSF Grant MCS-8201029.

Theorem 1.1 will be derived as an easy consequence of the following

1.2. THEOREM. *Suppose that* f: $X_0 \to X_1$ *is an n-connected map and that Y is* $(n+1)$*-co-connected, where* X_0, X_1, *and Y are connected spaces in* $\underline{W}$. *Then, each homotopy-fibre of the induced map* f^*: $\mathrm{map}(X_1, Y) \to \mathrm{map}(X_0, Y)$ *is either empty or contractible. If Y is n-co-connected, then no homotopy-fibre is empty.*

The n-connectivity assumption on f means that f_*: $\pi_i X_0 \to \pi_i X_1$ is injective for $i < n$ and surjective for $i \leqslant n$. The homotopy-fibre $\Phi(h; z)$ of a (not necessarily surjective) map h: $Z_0 \to Z_1$, with $z \in Z_1$, is the fibre over z in the fibration canonically associated with h. For the reader's convenience, we give the explicit definition of $\Phi(h; z)$, together with related facts, in §2.

1.3. REMARK. For spaces X, Y in $\underline{W}$ with nondegenerate basepoints (see 2.5), the function space $\mathrm{map}_*(X, Y)$ of basepoint-preserving maps $X \to Y$ is frequently used, rather than $\mathrm{map}(X, Y)$. Both Theorems 1.1 and 1.2 apply, as stated, to map_* in place of map. The proofs are virtually the same as the ones we shall give for the unpointed case. Alternatively, one can easily deduce the pointed results from the unpointed ones by making use of the "evaluation-fibration" $\mathrm{map}(X, Y) \to Y$, in which $\mathrm{map}_*(X, Y)$ is the fibre over the basepoint of Y (cf. 2.5).

Theorem 1.1 extends a classical result of J. Milnor [**9**], which implies that $\mathrm{map}(X, Y)$ belongs to $\underline{W}$ whenever Y belongs to $\underline{W}$ and X has the homotopy type of a finite complex. Although this result is extremely useful, the finiteness condition on X is troublesome. On the one hand, it cannot be avoided entirely [**9**, p. 273]. On the other hand, the condition is frequently violated by objects arising naturally in homotopy theory: e.g., Eilenberg-MacLane spaces $K(\pi, n)$, stages of a Postnikov tower, etc. Theorem 1.1 was motivated by such examples and can be applied to them in some cases.

To begin with, choose some Y in $\underline{W}$ of type $K(G, m)$, $m \geqslant 1$, and observe that Y is $(m+1)$-co-connected.

1.4. COROLLARY. *Suppose that X is a connected space in* $\underline{W}$ *of m-finite type. Then* (a) $\mathrm{map}(X, Y)$ *belongs to* $\underline{W}$, *and* (b) *if G is abelian, then* $\mathrm{map}(X, Y)$ *is homotopy equivalent to a product* $\prod_{i=0}^{m} Y_i$, *in which* Y_i *is a space in* $\underline{W}$ *of type* $K(H^{m-i}(X; G), i)$. *For example, when* $m = 1$, $\mathrm{map}(X, Y) \simeq H^1(X; G) \times Y$.

Assertion 1.4(a) is an immediate consequence of Theorem 1.1. For assertion 1.4(b), a computation of Thom [**15**, p. 31], shows that $\mathrm{map}(X, Y)$ has the stated singular homotopy type, and 1.4(a) implies that this is the same as its homotopy type.

As a special case, take X to be any $K(\pi, n)$ space in $\underline{W}$ satisfying one of the following conditions.

1.5. (1) $m < n$;

(2) π finitely-generated, abelian;

(3) π finitely-generated and $m = n = 1$;

(4) π finitely-presented and of type FP_m (cf. [**1**]), and $m > n = 1$.

Then X has m-finite type, and Corollary 1.4 may be applied to it.

Corollary 1.4 has a pointed analogue, which we do not state in general but instead specialize to a case that may have some independent geometric interest.

1.6. COROLLARY. *Let M be a (finite-dimensional, topological) manifold of type $K(\pi, 1)$ with $\pi = \pi_1 M$ finitely-generated, and let $G_*(M)$ denote the monoid of basepoint-preserving self-equivalences of M. Then each component of $G_*(M)$ is contractible.*

Note that M belongs to $\underline{W}$ and has nondegenerate basepoints. Thus, since 1.5(3) is satisfied, we may apply the pointed version of 1.4(a) (or of 1.1) to conclude that $G_*(M)$ is in $\underline{W}$. It is straightforward to compute that each component of $G_*(M)$ has trivial homotopy groups, and so Whitehead's Theorem then gives contractibility.

1.7. REMARKS. (a) The compact-open topology on $G_*(M)$ is metrizable, hence already compactly-generated.

(b) When M has the homotopy type of a compact manifold, Corollary 1.6 is known, proved simply by using Milnor's theorem in the above argument. However, see the next remark.

(c) There are numerous interesting examples of $K(\pi, 1)$ manifolds. For example, these arise naturally as Riemannian manifolds of constant negative curvature, or as classical knot complements. In fact, for every countable group π of finite cohomological dimension, there is a corresponding $K(\pi, 1)$ manifold [**18**, p. 320]. Many of these examples do not have the homotopy type of a finite complex, even when π is finitely-presented.

We now apply Theorem 1.1 to Postnikov stages. Note that if a space X in $\underline{W}$ has n-finite type, then so does $X_{(n)}$, its nth Postnikov stage. Thus, we have

1.8. COROLLARY. *Suppose that X is a connected space in $\underline{W}$ having n-finite type. Then, both* $\operatorname{map}(X_{(i)}, X_{(i)})$ *and* $G(X_{(i)})$, *the monoid of (unpointed) self-equivalences of $X_{(i)}$, belong to $\underline{W}$ for all $i \leqslant n$.* □

For our final application, let X be a connected space in $\underline{W}$ of n-finite type, and let $p_i\colon X_{(i)} \to X_{(i-1)}$ denote the ith map in a Postnikov tower for X, $i \leqslant n$. Then p_i induces a diagram:

$$(1.9)\qquad \begin{array}{ccccc} & & \operatorname{map}(X_{(i)}, X_{(i-1)}) & & \\ & \overset{(p_i)_*}{\nearrow} & & \overset{(p_i)^*}{\nwarrow} & \\ \operatorname{map}(X_{(i)}, X_{(i)}) & & & & \operatorname{map}(X_{(i-1)}, X_{(i-1)}) \\ & \supset & & \subset & \\ & & G(X_{(i)}) \dashrightarrow G(X_{(i-1)}) & & \end{array}$$

1.10. COROLLARY. (a) *There exists a map $\alpha\colon G(X_{(i)}) \to G(X_{(i-1)})$ completing* (1.9) (*i.e., making it commute up to homotopy*).

(b) *Such an α is unique, up to homotopy.*

(c) *α is an A_∞-map of monoids (in the sense of* [**13**]).

Theorem 1.1 implies that each of the spaces of (1.9) belongs to $\underline{W}$. Theorem 1.2 then implies that $(p_i)^*$ is a homotopy equivalence. This immediately gives 1.10(a), (b). The proof of (c) is given in §4.

1.11. REMARKS. In the category of simplicial sets, there is a canonical and functorial Postnikov-tower-construction, due to J. Moore (e.g., see [8]), which yields almost immediately the existence of a simplicial analogue of α. This analogue is a homomorphism of simplicial monoids.

In the topological category, the Postnikov-tower-construction is neither canonical nor functorial, so the existence of α cannot be established so readily.

If we apply the topological realization functor to the simplicial analogue mentioned above, we do obtain a suitable α, *provided that we know that* $G(X_{(i)})$ *has CW type*, which requires Theorem 1.1. Thus, this gives an alternative proof of 1.10(a). A third proof will arise (also depending on 1.1) in the process of demonstrating 1.10(c) in §4.

Note that 1.10(c) is as close as one can reasonably expect to get to asserting that α is a homomorphism of monoids.

2. Some technical facts. This section presents the technical information needed for the proofs of the theorems and of Corollary 1.10(c).

2.1. DEFINITIONS. Given a map $h\colon Z_0 \to Z_1$, set

$$E_h = \{(z, \omega) \in Z_0 \times \text{map}(I, Z_1) | \omega(0) = h(z)\},$$

and define $p_h\colon E_h \to Z_1$ by $p_h(z, \omega) = \omega(1)$. Here I is the unit interval $[0, 1]$. $Z_0 \times \text{map}(I, Z_1)$ and E_h are topologized in accordance with the usual conventions concerning the k-topology (cf. [14]). In particular, the k-topology on E_h coincides with the relative topology induced by the k-space $Z_0 \times \text{map}(I, Z_1)$.

In the classical context (i.e., in which the compact-open topology is used, etc.), p_h is a Hurewicz fibration (e.g., cf. [11, pp. 99–100]). It follows easily that if we then impose k-topologies, p_h becomes a fibration in the sense we describe in the introduction. Note that image p_h is the union of all path-components of Z_1 meeting image h.

The homotopy-fibre $\Phi(h; z_1)$ is defined to be $p_h^{-1}(z_1)$, for each $z_1 \in Z_1$.

We use " $\simeq$ " to denote homotopy-equivalence.

2.2. LEMMA. *If* $p\colon E \to B$ *is a fibration and* $b \in B$, *then* $p^{-1}(b) \simeq \Phi(p; b)$.

2.3. LEMMA. *Suppose that*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & C \\ \gamma\downarrow & & \downarrow\delta \\ B & \xrightarrow[\beta]{} & D \end{array}$$

is a homotopy-commutative diagram in which α *and* β *are homotopy-equivalences. Then, for every* $x \in B$, $\Phi(\gamma; x) \simeq \Phi(\delta; \beta(x))$.

Both lemmas are folklore in homotopy theory. The Eckmann-Hilton dual of a special case of 2.3 is proved in [6], and another special case follows from results in [10].

2.4. LEMMA. *Suppose that* $p: E \to B$ *is a fibration and* $B \in \underline{W}$. *Then* $E \in \underline{W}$ *if and only if each* $p^{-1}(b) \in \underline{W}$.

Except for our use of the k-topology, this lemma follows immediately from results of Stasheff [**12**] (more precisely, from his Propositions (0) and (12), together with our Lemma 2.2). Stasheff's arguments remain correct in our k-topology-context; in fact, they may be somewhat abbreviated here.

Note that these lemmas do not require maps to be surjective.

2.5. DEFINITIONS. (a) Suppose that $f: A \to X$ is a closed cofibration in $\underline{W}$, and consider the induced map f^*: map$(X, Y) \to$ map(A, Y), for any Y in $\underline{W}$. In the classical case (i.e., compact-open topology), this is a Hurewicz fibration [**11**, p. 97]. After imposing the k-topology, it becomes a fibration in our sense. We denote the fibre over $g \in$ map(A, Y) by map$(X, Y; g)$. When f is an inclusion map, we may think of map$(X, Y; g)$ as consisting of all extensions $\bar{g}: X \to Y$ of g.

Note that if A is a point, so that, by definition, f maps onto a nondegenerate basepoint $*$, then f^* may be identified with the evaluation map map$(X, Y) \to Y$, which sends $\bar{g}$ to $\bar{g}(*)$.

(b) Using the above notation, suppose that (X, A) is a CW pair, $A \neq \varnothing$, and f is the inclusion map (henceforth suppressed). As above, choose some $g: A \to Y$ and some extension $\bar{g} \in$ map$(X, Y; g)$. Finally, pick basepoints $x \in A \subseteq X$ and $y = g(x) \in Y$. Then, $\bar{g}$ determines the structure of a $\mathbf{Z}\pi_1(X, x)$-module on $\pi_m(Y, y)$, for any $m \geqslant 2$, which we denote by $\bar{g}^{\#}\pi_m Y$.

In Lemma 2.6, i is any fixed nonnegative integer, and cohomology will be taken with local coefficients.

2.6. LEMMA. *Assume the context of* 2.5(b), *and, in addition, suppose* (a) *the inclusion* $A \hookrightarrow X$ *is* 1*-connected, and* (b) $H^n(X, A; \bar{g}^{\#}\pi_{n+i}Y) = 0$, *for all* $n + i \geqslant 2$. *Then*, $\pi_i(\text{map}(X, Y; g)) = 0$, *where the homotopy group* π_i *is based at* $\bar{g}$ *when* $i \geqslant 1$.

PROOF. A typical element of $\pi_i(\text{map}(X, Y; g))$ is represented by a map ϕ: $S^i \to$ map$(X, Y; g)$, based at $\bar{g}$, which we may transform, by adjointness to a map $\tilde{\phi}$: $S^i \times X \to Y$. The conclusion $\pi_i = 0$ is then seen to be equivalent to the existence, for each ϕ, of an extension ψ in the diagram

$$\begin{array}{ccc} S^i \times X \cup D^{i+1} \times A & \xrightarrow{\tilde{\phi} \cup g \circ \mathrm{pr}_A} & Y \\ j \big\downarrow & \overset{\psi}{\nearrow} & \\ D^{i+1} \times X & & \end{array}$$

where pr_A: $D^{i+1} \times A \to A$ is the canonical projection. The hypotheses imply that j induces a π_1-isomorphism, so that ψ may be defined on the 2-skeleton. Further obstructions to this extension lie in zero groups. □

For each $i \geqslant 0$, define the relative i-skeleton $\overline{X}^i$ to be $A \cup \bigcup\{$cells of X of dimension $\leqslant i\}$. The inclusions $\overline{X}^i \subseteq \overline{X}^{i+1}$ induce maps map$(\overline{X}^{i+1}, Y; g) \to$ map$(\overline{X}^i, Y; g)$ making $\{\text{map}(\overline{X}^i, Y; g)\}$ into an inverse limit system.

2.7. LEMMA. *Restriction induces a homeomorphism*

$$\operatorname{map}(X, Y; g) \approx \varprojlim_i \operatorname{map}(\overline{X}^i, Y; g).$$

The map induced by restriction is easily seen to be continuous and bijective, and so the burden of proof is to check that it is open. We leave this to the reader.

2.8. LEMMA. *If, in the tower*

$$* \xleftarrow{p_1} E_1 \xleftarrow{p_2} E_2 \xleftarrow{p_3} E_3 \leftarrow \cdots$$

of fibrations, each $E_i \simeq *$, *then* $\varprojlim E_i \simeq *$.

PROOF. Because each $E_i \simeq *$, each p_i is surjective, so that $\varprojlim E_i \neq \varnothing$. Choose $\{e_i\} \in \varprojlim E_i$, and set $F_i = p_i^{-1}(e_{i-1}) \subseteq E_i$. Because the fibrations are fibre-homotopy trivial, each $F_i \simeq *$.

We define a family of contractions h_i: $E_i \times [0, 1] \to E_i$, $i \geqslant 1$, satisfying (i) $h_i(x, t) = e_i$, for all $(x, t) \in E_i \times [0, 2^{-i}]$, and (ii) $p_i h_i = h_{i-1}(p_i \times \mathrm{id}_{[0,1]})$. By the universal property for $\varprojlim$ (in the category of k-spaces), the h_i fit together to give a contraction of $\varprojlim E_i$. More precisely, $\varprojlim h_i$ is a continuous map $\varprojlim(E_i \times [0, 1]) \to \varprojlim E_i$, which, when composed with the natural homeomorphism

$$\left(\varprojlim E_i\right) \times [0, 1] \to \varprojlim (E_i \times [0, 1]),$$

yields the desired contraction. It remains to define the h_i.

Define h_1 to be any contraction satisfying (i) for $i = 1$, and suppose h_{j-1} is defined satisfying (i), (ii) for $i = j - 1$. Use homotopy-lifting to obtain a map h'_j: $E_j \times [0, 1] \to E_j$ such that h'_j satisfies (ii) for $i = j$, and $h'_j(x, 1) = x$, for all $x \in E_j$. Clearly, $h'_j(E_j \times [0, 2^{-j+1}]) \subseteq F_j$. Use the contractibility of F_j to define a deformation d_t in F_j between the trivial map $E_j \to \{e_j\}$ and $h'_j | E_j \times \{2^{-j+1}\}$. Then, define h_j as follows:

$$h_j(x, t) = \begin{cases} e_j, & 0 \leqslant t \leqslant 2^{-j}, \\ d_s(x), & 2^{-j} \leqslant t \leqslant 2^{-j+1}, s = 2^j(t - 2^{-j}), \\ h'_j(x, t), & 2^{-j+1} \leqslant t \leqslant 1. \end{cases}$$

Clearly, h_j is a contraction satisfying (i), (ii) for $i = j$. □

This completes the technical information needed for our proofs of Theorems 1.1 and 1.2. The remaining facts will be used for our proof of Corollary 1.10(c).

We shall make use of the Moore loop space ΛZ of a space Z, which has the same homotopy type as the ordinary loop space and has a strictly associative loop composition (cf. [**13**, p. 14]).

Consider the self-equivalence monoid $G(X)$ of a space X in $\underline{W}$, and let $BG(X)$ denote the classifying space constructed by Dold and Lashof [**3**].

2.9. FACT. If $G(X)$ is in $\underline{W}$, then there is an A_∞-equivalence of monoids $G(X) \approx \Lambda BG(X)$.

The Dold-Lashof construction produces a weak homotopy equivalence $G(X) \to \Lambda BG(X)$ (see [**13**, Theorem 4.3]). Since $G(X)$ is in $\underline{W}$, an argument of Stasheff [**12**, p. 243], implies that $BG(X)$ is in $\underline{W}$, and then Milnor's theorem yields the same for $\Lambda BG(X)$. It follows that the weak equivalence is a homotopy equivalence. Work of Stasheff and of Fuchs (cf. [**13**, pp. 33–35]) then shows that this homotopy equivalence is an A_∞-equivalence of monoids.

2.10. FACT. Suppose that X and $G(X)$ are in $\underline{W}$, as above. Then $BG(X)$ classifies X-fibrations over spaces in $\underline{W}$.

Stasheff [**12**] proves such a result, without using k-topologies, under the assumption that X is a finite complex. The finiteness assumption insures that certain maps in Stasheff's constructions are continuous and that $G(X)$ belongs to $\underline{W}$, and this is its only use. In our case, the k-topology will insure continuity and $G(X) \in \underline{W}$ by hypothesis. Thus, Stasheff's argument yields 2.10.

An alternative proof follows from Dold's representability theorem [**2**], obtaining a space $B(X)$ in $\underline{W}$ which classifies X-fibrations. The assumption $G(X) \in \underline{W}$ is then needed to verify that $B(X) \simeq BG(X)$.

3. Proofs of the theorems.

3.1. Theorem 1.2 $\Rightarrow$ Theorem 1.1. We may take X to be a CW complex with finite n-skeleton, X^n, and we let $f\colon X^n \to X$ be the inclusion. Then $f^*\colon \operatorname{map}(X, Y) \to \operatorname{map}(X^n, Y)$ is a fibration. By 1.2, together with 2.2, the fibres of f^* are either empty or contractible: in either case they belong to $\underline{W}$. Since $\operatorname{map}(X^n, Y) \in \underline{W}$, by Milnor's theorem, Lemma 2.4 now gives the desired conclusion. □

3.2. PROOF OF THEOREM 1.2. The proof proceeds in a number of steps.

3.2.1. We observe first that we may assume that (X_1, X_0) is a CW pair with X_0 containing the n-skeleton of X_1 and f the inclusion $X_0 \to X_1$. For in the general case, one can always find such a CW pair (K_1, K_0) and a homotopy-commutative square

$$\begin{array}{ccc} X_0 & \overset{a_0}{\to} & K_0 \\ f\downarrow & & \hookdownarrow \\ X_1 & \underset{a_1}{\to} & K_1 \end{array}$$

in which each a_i is a homotopy-equivalence. Apply map $(-, Y)$ to this square, and then apply Lemma 2.3 to the result.

Henceforth, we assume that the above reduction has been made. Moreover, in light of Lemma 2.2, it suffices to verify the conclusions of Theorem 1.2 for each honest fibre of the fibration $f^*\colon \operatorname{map}(X_1, Y) \to \operatorname{map}(X_0, Y)$. Finally, we may suppose that $n > 0$, because when $n = 0$, Y is contractible, and the result is trivially true.

3.2.2. *Case* 1. $X_1 = X_0 \cup_\phi D^p$, $p > n$. Choose $a \in \operatorname{map}(X_0, Y)$. When Y is n-coconnected, a extends over X_1, so that $(f^*)^{-1}(a) = \operatorname{map}(X_1, Y; a)$ is then nonempty. In any case, $\operatorname{map}(X_1, Y; a)$ is homeomorphic to $\operatorname{map}(D^p, Y; a\phi)$, which we henceforth denote by M, and which is the fibre over $a\phi$ in the restriction-fibration $\operatorname{map}(D^p, Y) \to \operatorname{map}(S^{p-1}, Y)$. By Milnor's theorem and Lemma 2.4, this fibre M

belongs to $\underline{W}$, so that, if it is nonempty, we need only show that $\pi_i M = 0$, for all $i \geqslant 0$. But this follows from Lemma 2.6, together with the $(n+1)$-co-connectivity of Y, which completes the proof in this case.

3.2.3. *Case* 2. $X_1 = X_0 \cup_\phi \sqcup_\alpha D^p_\alpha$, $p > n$. Set $\phi_\alpha = \phi | S^{p-1}_\alpha$. Then, $(f^*)^{-1}(a)$ is homeomorphic to $\prod_\alpha \text{map}(D^p_\alpha, Y; a\phi_\alpha)$ (cf. Lemma 2.7). Case 1 shows that each factor in this product is empty or contractible. Hence, so is $(f^*)^{-1}(a)$.

3.2.4. *Case* 3. (X_1, X_0) is a CW pair with cells in $X_1 \setminus X_0$ having bounded dimension $> n$. We have a filtration of X_1 by relative skeleta (cf. §2)

$$X_0 = \bar{X}^n \subset \bar{X}^{n+1} \subset \cdots \subset \bar{X}^q = X_1,$$

for some $q \leqslant n$, $q < \infty$. Apply map$(-, Y)$ to this filtration, obtaining a finite tower of fibrations in which each fibre is empty or contractible, either because of 3.2.3 or because it is a point. The total fibre over a, $(f^*)^{-1}(a)$, if nonempty, is itself then fibred

$$* = \text{map}(\bar{X}^n, Y; a) \leftarrow \text{map}(\bar{X}^{n+1}, Y; a) \leftarrow \cdots \leftarrow \text{map}(\bar{X}^q, Y; a) = (f^*)^{-1}(a)$$

by a tower of fibrations whose fibres are selected from the (nonempty) fibres above. Hence, they are contractible, and an easy induction shows that $(f^*)^{-1}(a)$ is contractible.

3.2.5. *Case* 4. (X_1, X_0) is a CW pair such that X_0 contains the n-skeleton of X_1. When $(f^*)^{-1}(a)$ is nonempty, use 3.2.4 to obtain a (possibly infinite) tower of fibrations

$$* = \text{map}(\bar{X}^n, Y; a) \leftarrow \text{map}(\bar{X}^{n+1}, Y; a) \leftarrow \cdots$$

in which each total space is contractible. The desired result now follows immediately from Lemmas 2.7 and 2.8.

In light of 3.2.1, the proof of Theorem 1.2 is now complete.

4. Proof of Corollary 1.10(c). Recall that X is a connected space in $\underline{W}$ of n-finite type and $X_{(j)}$ denotes its jth Postnikov stage. When $j \leqslant n$, $G(X_{(j)})$ belongs to $\underline{W}$ (Corollary 1.8), so that we may make use of Facts 2.9 and 2.10. To begin with, we shall identify $G(X_{(j)})$ with $\Lambda BG(X_{(j)})$ via the A_∞-equivalence of 2.9.

Now let $i \leqslant n$ be as in the statement of 1.10. The proof proceeds by defining a map

$$BG(X_{(i)}) \xrightarrow{\beta} BG(X_{(i-1)})$$

such that $\Lambda\beta$ completes (1.9) (thus giving a third proof of Corollary 1.10(a)). $\Lambda\beta$ is, of course, an A_∞-map of monoids.

Any map α: $G(X_{(i)}) \to G(X_{(i-1)})$ completing (1.9) is homotopic to $\Lambda\beta$, by Corollary 1.10(b), and this finishes the proof, because A_∞-maps of monoids are closed under homotopy [**13**, p. 33].

It remains to define β with the requisite property. By 2.10, there is a universal $X_{(i)}$-fibration $EG(X_{(i)}) \to BG(X_{(i)})$. Its $(i-1)$st Moore-Postnikov stage is an $X_{(i-1)}$-fibration over $BG(X_{(i)})$, which may be classified by a map $BG(X_{(i)}) \to BG(X_{(i-1)})$ because $BG(X_{(i)})$ is in $\underline{W}$. This classifying map is β.

The argument that $\Lambda\beta$ completes (1.9) has two steps, whose details we leave to the reader: (1) if $f\colon B \to BG(X_{(i)})$ classifies the $X_{(i)}$-fibration $p\colon E \to B$, verify that $\beta \circ f$ classifies the $(i-1)$st Moore-Postnikov stage of p. (2) When B is a suspension ΣA, the fibration $p\colon E \to B$ is classified by a "clutching function" $g\colon A \to G(X_{(i)})$. Using (1), together with standard arguments, verify that the $(i-1)$st Moore-Postnikov stage of p is then classified by $\Lambda\beta \circ g$. This readily implies that $\Lambda\beta$ completes (1.9) and concludes our proof.

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