TWO RESULTS CONCERNING CARDINAL FUNCTIONS ON COMPACT SPACES

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ABSTRACT. We show that for $X$ compact $T_2$: (i) $d(X) \leq s(X) \cdot \hat{F}(X)$; (ii) if the pair $(\kappa, \hat{F}(X))$ is a caliber of $X$ then $\pi(X) < \kappa$.

These strengthen results of Šapirovskii from [3 and 5], respectively. Moreover, (i) settles a problem raised in [2] implying that there are no compact $T_2 \kappa$-examples for any singular cardinal $\kappa$.

In this note we follow the notation and terminology of [1]. In particular, we let $\hat{F}(X)$ denote the smallest cardinal $\kappa$ such that $|S| < \kappa$ for any free sequence $S \subseteq X$.

**THEOREM 1.** If $X$ is compact $T_2$ then $d(X) \leq s(X) \cdot \hat{F}(X)$.

**Proof.** Let us put $s(X) \cdot \hat{F}(X) = \kappa$. Given any nonempty open set $U \subseteq X$ we can choose a family $\hat{C}(U)$ of open $F_\alpha$ sets in $X$ such that $U = \bigcup \hat{C}(U)$. But $X$ does not contain discrete subspaces of cardinality $\kappa^+$, hence, e.g. by 2.13 of [1], there is a subfamily $\hat{B}(U) \subseteq \hat{C}(U)$ and a subset $S(U) \subseteq U$ such that $|\hat{B}(U)| \leq \kappa$, $|S(U)| \leq \kappa$ and $U \subseteq \bigcup \hat{B}(U) \cup S(U)$.

Let us now assume, indirectly, that $d(X) > \kappa$. Then we also have $\pi(X) > \kappa$. Hence if $\mathcal{A}$ is a family of nonempty closed $G_\delta$ sets in $X$ with $|\mathcal{A}| \leq \kappa$, then there is an open nonempty $U \subseteq X$ such that $A \setminus U \neq \emptyset$ for each $A \in \mathcal{A}$. It follows easily from the compactness of $X$ that if $\mathcal{U}$ is a chain of open sets with this property, then $\bigcup \mathcal{U}$ possesses it as well. Thus by Zorn’s lemma, we can fix an open set $W(\mathcal{A})$ which is maximal with respect to the above property. Observe that then for every nonempty set $H$ open in $X \setminus W(\mathcal{A})$, there is an $A \in \mathcal{A}$ with $A \subseteq H \cup W(\mathcal{A})$. Hence $\emptyset \neq A \setminus W(\mathcal{A}) \subset H$, i.e. $\{A \setminus W(\mathcal{A}) : A \in \mathcal{A}\}$ is a $\pi$-network in $X \setminus W(\mathcal{A})$. Consequently, we have

$$d(X \setminus W(\mathcal{A})) \leq |\mathcal{A}| \leq \kappa.$$ 

After these preparations we define by transfinite induction, families $\mathcal{B}_\alpha$ of closed $G_\delta$ subsets of $X$ for $\alpha \in \kappa$ with $|\mathcal{B}_\alpha| \leq \kappa$ as follows. If $\alpha \in \kappa$ and $\mathcal{B}_\beta$ has been defined for all $\beta < \alpha$, we consider the open set $W_\alpha = W(\bigcup \{\mathcal{B}_\beta : \beta < \alpha\})$ and the family $\mathcal{B}(W_\alpha)$ of open $F_\alpha$ subsets of $W_\alpha$. For every $G \in \mathcal{B}(W_\alpha)$ we may then choose closed $G_\delta$ sets $F_G^n$ for $n \in \omega$ such that $G = \bigcup \{F_G^n : n \in \omega\}$. $\mathcal{B}_\alpha$ is then defined as the set of all nonempty finite intersections of members of the family

$$\bigcup \{\mathcal{B}_\beta : \beta < \alpha\} \cup \{X \setminus G : G \in \mathcal{B}(W_\alpha)\} \cup \{F_G^n : G \in \mathcal{B}(W_\alpha), n \in \omega\}.$$ 

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Clearly \(|\mathcal{B}_\alpha| \leq \kappa\). This completes the induction.

Let us put

\[ Y = \bigcup \{ S(W_\alpha) \cup (X \setminus W_\alpha) : \alpha \in \kappa \}; \]

since \(|S(W_\alpha)| \leq \kappa\) and \(d(X \setminus W_\alpha) \leq \kappa\), then \(d(Y) \leq \kappa\) as well; hence, by our indirect assumption, \(X \neq Y\).

We may thus pick a point \(p \in X \setminus Y\). Then for every \(\alpha \in \kappa\) there are \(G_\alpha \in \beta(W_\alpha)\) and \(n_\alpha \in \omega\) such that \(p \in F_\alpha = F_{G_\alpha} \subset G_\alpha\). Let us put, for \(\alpha \in \kappa\),

\[ \mathcal{L}_\alpha = \{ F_\beta : \beta \leq \alpha \} \cup \{ X \setminus G_\beta : \alpha < \beta < \kappa \}. \]

We claim that \(\mathcal{L}_\alpha\) is centered. Indeed, if \(\mathcal{Z} \in [\mathcal{L}_\alpha]^\omega\), then \(\bigcap \mathcal{Z} \neq \emptyset\) follows by an easy induction on the number of the \(X \setminus G_\beta\)'s in \(\mathcal{Z}\) from our above construction.

Since the members of \(\mathcal{L}_\alpha\) are closed and nonempty, we can pick for each \(\alpha \in \kappa\) a point \(p_\alpha \in \bigcap \mathcal{L}_\alpha\). But then \(\{ p_\beta : \beta \leq \alpha \} \subset X \setminus G_\alpha\) and \(\{ p_\beta : \beta \in \kappa \setminus \alpha \} \subset F_\alpha\); hence

\[ \bigcap \{ p_\beta : \beta \leq \alpha \} \cap \bigcap \{ p_\beta : \beta \in \kappa \setminus \alpha \} = \emptyset. \]

This, however is a contradiction since \(\{ p_\alpha : \alpha \in \kappa \}\) is a free sequence in \(X\) of size \(\kappa \geq \hat{F}(X)\). Hence our proof is completed.

Theorem 1 is a strengthening of Šapirovskii's result saying that

\[ d(X) \leq s(X) \cdot t(X)^+ \]

for \(X\) compact \(T_2\), since, as is well known (see e.g. [1, 3.12]), for \(X\) compact \(T_2\) we have \(F(X) = t(X)\). However the proofs of this given in [3, 4 or 1] do not seem to be modifiable to yield our result for the case in which \(s(X) \cdot \hat{F}(X) = \hat{F}(X) = \kappa\) is a singular cardinal. That this case is of some independent interest is shown by the following result that solves a problem raised in [2] (and answered there only partially even for the case \(\text{cf}(\kappa) \leq \omega_1\)).

**Corollary.** If \(X\) is compact \(T_2\), \(\kappa\) is a singular cardinal, and \(\pi(Y) < \kappa\) holds for each subspace \(Y \subset X\) with \(|Y| \leq \kappa\), then \(\pi(X) < \kappa\) as well (or in the terminology of [2] there are no compact \(T_2\) \(\kappa\)-examples).

**Proof.** Clearly \(X\) may have no discrete subspaces of cardinality \(\kappa\). Hence we have \(d(X) \leq s(X) \cdot \hat{F}(X) \leq \kappa\). But if \(Y \subset X\) is dense with \(|Y| \leq \kappa\), then by our assumption and 2.7 of [1], \(\pi(X) = \pi(Y) < \kappa\).

To formulate our next result we recall that a pair \(\langle \kappa, \lambda \rangle\) of cardinals is said to be a caliber of a space \(X\) if for every family \(\{ G_\xi : \xi \in \kappa \}\) of nonempty open sets in \(X\) there is a set \(A \subset \kappa\) with \(|A| = \lambda\) such that \(\bigcap \{ G_\xi : \xi \in A \} \neq \emptyset\).

**Theorem 2.** If \(X\) is compact \(T_2\) and the pair \(\langle \kappa, \hat{F}(X) \rangle\) is a caliber of \(X\), then \(\pi(X) < \kappa\).

**Proof.** Since the proof is quite similar to, but actually even simpler than, that of Theorem 1, we give only a sketch.

First, for any nonempty open set \(U \subset X\) we fix a family \(\mathcal{C}(U)\) of open \(F_\alpha\)'s in \(X\) whose union is \(U\). Second, assuming indirectly that \(\pi(X) \geq \kappa\) and using that \(\kappa > \omega\), for any family \(\mathcal{A}\) of nonempty closed \(G_\delta\)'s with \(|\mathcal{A}| < \kappa\), we pick a nonempty open \(F_\alpha\).
set \( W(\cdot) \) such that \( A \setminus W(\cdot) \neq \emptyset \) for all \( A \in \mathcal{C} \). Then, by transfinite induction, families \( \mathcal{B}_\alpha \) of nonempty closed \( G_\delta \) sets with \( |\mathcal{B}_\alpha| \leq |\alpha| + \omega < \kappa \) are defined for \( \alpha \in \kappa \) as follows. If \( \alpha \in \kappa \) and \( \mathcal{B}_\beta \) have been chosen for \( \beta \in \alpha \), put

\[
W_\alpha = W\left( \bigcup \{ \mathcal{B}_\beta : \beta \in \alpha \} \right).
\]

We can write

\[
W_\alpha = \bigcup \{ F_\alpha^n : n \in \omega \},
\]

where the \( F_\alpha^n \) are closed \( G_\delta \) sets in \( X \). Now, \( \mathcal{B}_\alpha \) is defined as the set of all nonempty finite intersections of members of the family

\[
\bigcup \{ \mathcal{B}_\beta : \beta \in \alpha \} \cup \{ X \setminus W_\alpha \} \cup \{ F_\alpha^n : n \in \omega \}.
\]

Clearly, \( |\mathcal{B}_\alpha| \leq |\alpha| + \omega \).

Considering the family \( \{ W_\alpha : \alpha \in \kappa \} \) and using that, with \( \lambda = \hat{F}(X) \), the pair \( \langle \kappa, \lambda \rangle \) is a caliber of \( X \), we can find a set \( A \subset \kappa \) with \( |A| = \lambda \) such that

\[
\bigcap \{ W_\alpha : \alpha \in A \} \neq \emptyset.
\]

Let \( p \in \bigcap \{ W_\alpha : \alpha \in A \} \), and for each \( \alpha \in A \) choose \( n_\alpha \in \omega \) such that \( p \in F_\alpha^{n_\alpha} = F_\alpha \subset W_\alpha \). Exactly as in the proof of Theorem 1 we can see that for \( \alpha \in A \) the family

\[
\mathcal{L}_\alpha = \{ F_\beta : \beta \in A \& \beta \leq \alpha \} \cup \{ X \setminus W_\beta : \beta \in A \& \alpha < \beta \}
\]

is centered, and if \( p_\alpha \in \bigcap \mathcal{L}_\alpha \) for \( \alpha \in A \), then \( \{ p_\alpha : \alpha \in A \} \) is a free sequence in \( X \) of size \( \lambda = \hat{F}(X) \), a contradiction. This completes the proof.

In [5] Šapirovskii proved that if \( t(X)^+ \) is a caliber of a compact \( T_2 \) space \( X \) then \( \pi(X) \leq t(X) \). Since \( \hat{F}(X) \leq F(X)^+ = t(X)^+ \) and, moreover, if \( F(X)^+ \) is a caliber of \( X \), clearly so is the pair \( \langle F(X)^+, \hat{F}(X) \rangle \) as well, this result is an immediate corollary of Theorem 2.

REFERENCES


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