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# TWO RESULTS CONCERNING CARDINAL FUNCTIONS ON COMPACT SPACES

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ABSTRACT. We show that for $X$ compact $T_2$: (i) $d(X) \leqslant s(X) \cdot \hat{F}(X)$; (ii) if the pair $(\kappa, \hat{F}(X))$ is a caliber of $X$ then $\pi(X) < \kappa$.

These strengthen results of Šapirovskii from [3 and 5], respectively. Moreover, (i) settles a problem raised in [2] implying that there are no compact $T_2$ $\kappa$-examples for any singular cardinal $\kappa$.

In this note we follow the notation and terminology of [1]. In particular, we let $\hat{F}(X)$ denote the smallest cardinal $\kappa$ such that $|S| < \kappa$ for any free sequence $S \subset X$.

THEOREM 1. *If $X$ is compact $T_2$ then $d(X) \leqslant s(X) \cdot \hat{F}(X)$.*

PROOF. Let us put $s(X) \cdot \hat{F}(X) = \kappa$. Given any nonempty open set $U \subset X$ we can choose a family $\mathcal{C}(U)$ of open $F_\sigma$ sets in $X$ such that $U = \bigcup \mathcal{C}(U)$. But $X$ does not contain discrete subspaces of cardinality $\kappa^+$, hence, e.g. by 2.13 of [1], there is a subfamily $\mathcal{B}(U) \subset \mathcal{C}(U)$ and a subset $S(U) \subset U$ such that $|\mathcal{B}(U)| \leqslant \kappa$, $|S(U)| \leqslant \kappa$ and $U \subset \bigcup \mathcal{B}(U) \cup \overline{S(U)}$.

Let us now assume, indirectly, that $d(X) > \kappa$. Then we also have $\pi(X) > \kappa$. Hence if $\mathcal{A}$ is a family of nonempty closed $G_\delta$ sets in $X$ with $|\mathcal{A}| \leqslant \kappa$, then there is an open nonempty $U \subset X$ such that $A \setminus U \neq \varnothing$ for each $A \in \mathcal{A}$. It follows easily from the compactness of $X$ that if $\mathcal{U}$ is a chain of open sets with this property, then $\bigcup \mathcal{U}$ possesses it as well. Thus by Zorn's lemma, we can fix an open set $W(\mathcal{A})$ which is maximal with respect to the above property. Observe that then for every nonempty set $H$ open in $X \setminus W(\mathcal{A})$, there is an $A \in \mathcal{A}$ with $A \subset H \cup W(\mathcal{A})$. Hence $\varnothing \neq A \setminus W(\mathcal{A}) \subset H$, i.e. $\{A \setminus W(\mathcal{A}): A \in \mathcal{A}\}$ is a $\pi$-network in $X \setminus W(\mathcal{A})$. Consequently, we have

$$d(X \setminus W(\mathcal{A})) \leqslant |\mathcal{A}| \leqslant \kappa.$$

After these preparations we define by transfinite induction, families $\mathcal{B}_\alpha$ of closed $G_\delta$ subsets of $X$ for $\alpha \in \kappa$ with $|\mathcal{B}_\alpha| \leqslant \kappa$ as follows. If $\alpha \in \kappa$ and $\mathcal{B}_\beta$ has been defined for all $\beta \in \alpha$, we consider the open set $W_\alpha = W(\bigcup \{\mathcal{B}_\beta: \beta \in \alpha\})$ and the family $\mathcal{B}(W_\alpha)$ of open $F_\sigma$ subsets of $W_\alpha$. For every $G \in \mathcal{B}(W_\alpha)$ we may then choose closed $G_\delta$ sets $F_G^n$ for $n \in \omega$ such that $G = \bigcup \{F_G^n: n \in \omega\}$. $\mathcal{B}_\alpha$ is then defined as the set of all *nonempty* finite intersections of members of the family

$$\bigcup \{\mathcal{B}_\beta: \beta \in \alpha\} \cup \{X \setminus G: G \in \mathcal{B}(W_\alpha)\} \cup \{F_G^n: G \in \mathcal{B}(W_\alpha), n \in \omega\}.$$

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Clearly $|\mathfrak{B}_\alpha| \leqslant \kappa$. This completes the induction.

Let us put

$$Y = \bigcup \left\{ \overline{S(W_\alpha)} \cup (X \setminus W_\alpha) \colon \alpha \in \kappa \right\};$$

since $|S(W_\alpha)| \leqslant \kappa$ and $d(X \setminus W_\alpha) \leqslant \kappa$, then $d(Y) \leqslant \kappa$ as well; hence, by our indirect assumption, $X \neq Y$.

We may thus pick a point $p \in X \setminus Y$. Then for every $\alpha \in \kappa$ there are $G_\alpha \in \mathfrak{B}(W_\alpha)$ and $n_\alpha \in \omega$ such that $p \in F_\alpha = F_{G_\alpha}^{n_\alpha} \subset G_\alpha$. Let us put, for $\alpha \in \kappa$,

$$\mathfrak{Z}_\alpha = \{F_\beta \colon \beta \leqslant \alpha\} \cup \{X \setminus G_\beta \colon \alpha < \beta < \kappa\}.$$

We claim that $\mathfrak{Z}_\alpha$ is centered. Indeed, if $\mathfrak{Z} \in [\mathfrak{Z}_\alpha]^{<\omega}$, then $\bigcap \mathfrak{Z} \neq \varnothing$ follows by an easy induction on the number of the $X \setminus G_\beta$'s in $\mathfrak{Z}$ from our above construction.

Since the members of $\mathfrak{Z}_\alpha$ are closed and nonempty, we can pick for each $\alpha \in \kappa$ a point $p_\alpha \in \bigcap \mathfrak{Z}_\alpha$. But then $\{p_\beta \colon \beta \in \alpha\} \subset X \setminus G_\alpha$ and $\{p_\beta \colon \beta \in \kappa \setminus \alpha\} \subset F_\alpha$; hence

$$\overline{\{p_\beta \colon \beta \in \alpha\}} \cap \overline{\{p_\beta \colon \beta \in \kappa \setminus \alpha\}} = \varnothing.$$

This, however is a contradiction since $\{p_\alpha \colon \alpha \in \kappa\}$ is a free sequence in $X$ of size $\kappa \geqslant \hat{F}(X)$. Hence our proof is completed.

Theorem 1 is a strengthening of Šapirovskii's result saying that

$$d(X) \leqslant s(X) \cdot t(X)^+$$

for $X$ compact $T_2$, since, as is well known (see e.g. [**1**, 3.12]), for $X$ compact $T_2$ we have $F(X) = t(X)$. However the proofs of this given in [**3**, **4** or **1**] do not seem to be modifiable to yield our result for the case in which $s(X) \cdot \hat{F}(X) = \hat{F}(X) = \kappa$ is a singular cardinal. That this case is of some independent interest is shown by the following result that solves a problem raised in [**2**] (and answered there only partially even for the case $\mathrm{cf}(\kappa) \leqslant \omega_1$).

COROLLARY. *If $X$ is compact $T_2$, $\kappa$ is a singular cardinal, and $\pi(Y) < \kappa$ holds for each subspace $Y \subset X$ with $|Y| \leqslant \kappa$, then $\pi(X) < \kappa$ as well (or in the terminology of* [**2**] *there are no compact $T_2$ $\kappa$-examples).*

PROOF. Clearly $X$ may have no discrete subspaces of cardinality $\kappa$. Hence we have $d(X) \leqslant s(X) \cdot \hat{F}(X) \leqslant \kappa$. But if $Y \subset X$ is dense with $|Y| \leqslant \kappa$, then by our assumption and 2.7 of [**1**], $\pi(X) = \pi(Y) < \kappa$.

To formulate our next result we recall that a pair $\langle \kappa, \lambda \rangle$ of cardinals is said to be a caliber of a space $X$ if for every family $\{G_\xi \colon \xi \in \kappa\}$ of nomempty open sets in $X$ there is a set $A \subset \kappa$ with $|A| = \lambda$ such that $\bigcap \{G_\xi \colon \xi \in A\} \neq \varnothing$.

THEOREM 2. *If $X$ is compact $T_2$ and the pair $\langle \kappa, \hat{F}(X) \rangle$ is a caliber of $X$, then $\pi(X) < \kappa$.*

PROOF. Since the proof is quite similar to, but actually even simpler than, that of Theorem 1, we give only a sketch.

First, for any nonempty open set $U \subset X$ we fix a family $\mathcal{C}(U)$ of open $F_\sigma$'s in $X$ whose union is $U$. Second, assuming indirectly that $\pi(X) \geqslant \kappa$ and using that $\kappa > \omega$, for any family $\mathcal{Q}$ of nonempty closed $G_\delta$'s with $|\mathcal{Q}| < \kappa$, we pick a nonempty open $F_\sigma$

set $W(\mathcal{Q})$ such that $A\setminus W(\mathcal{Q}) \neq \emptyset$ for all $A \in \mathcal{Q}$. Then, by transfinite induction, families $\mathfrak{B}_\alpha$ of nonempty closed $G_\delta$ sets with $|\mathfrak{B}_\alpha| \leqslant |\alpha| + \omega < \kappa$ are defined for $\alpha \in \kappa$ as follows. If $\alpha \in \kappa$ and $\mathfrak{B}_\beta$ have been chosen for $\beta \in \alpha$, put

$$W_\alpha = W\big(\bigcup \{\mathfrak{B}_\beta : \beta \in \alpha\}\big).$$

We can write

$$W_\alpha = \bigcup \{F_\alpha^n : n \in \omega\},$$

where the $F_\alpha^n$ are closed $G_\delta$ sets in $X$. Now, $\mathfrak{B}_\alpha$ is defined as the set of all nonempty finite intersections of members of the family

$$\bigcup \{\mathfrak{B}_\beta : \beta \in \alpha\} \cup \{X\setminus W_\alpha\} \cup \{F_\alpha^n : n \in \omega\}.$$

Clearly, $|\mathfrak{B}_\alpha| \leqslant |\alpha| + \omega$.

Considering the family $\{W_\alpha : \alpha \in \kappa\}$ and using that, with $\lambda = \hat{F}(X)$, the pair $\langle \kappa, \lambda \rangle$ is a caliber of $X$, we can find a set $A \subset \kappa$ with $|A| = \lambda$ such that

$$\bigcap \{W_\alpha : \alpha \in A\} \neq \emptyset.$$

Let $p \in \bigcap \{W_\alpha : \alpha \in A\}$, and for each $\alpha \in A$ choose $n_\alpha \in \omega$ such that $p \in F_\alpha^{n_\alpha} = F_\alpha \subset W_\alpha$. Exactly as in the proof of Theorem 1 we can see that for $\alpha \in A$ the family

$$\mathfrak{X}_\alpha = \{F_\beta : \beta \in A \;\&\; \beta \leqslant \alpha\} \cup \{X\setminus W_\beta : \beta \in A \;\&\; \alpha < \beta\}$$

is centered, and if $p_\alpha \in \bigcap \mathfrak{X}_\alpha$ for $\alpha \in A$, then $\{p_\alpha : \alpha \in A\}$ is a free sequence in $X$ of size $\lambda = \hat{F}(X)$, a contradiction. This completes the proof.

In [5] Šapirovskii proved that if $t(X)^+$ is a caliber of a compact $T_2$ space $X$ then $\pi(X) \leqslant t(X)$. Since $\hat{F}(X) \leqslant F(X)^+ = t(X)^+$ and, moreover, if $F(X)^+$ is a caliber of $X$, clearly so is the pair $\langle F(X)^+, \hat{F}(X)\rangle$ as well, this result is an immediate corollary of Theorem 2.

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