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A TOPOLOGICAL SPACE WITHOUT A COMPLETE QUASI-UNIFORMITY

HANS-PETER A. KÜNZI[1] AND PETER FLETCHER

ABSTRACT. We show that an example of Burke and van Douwen has no complete quasi-uniformity. Moreover, we show that it is almost finitely-fully normal but not almost $\aleph_0$-fully normal.

0. Introduction. Every topological space admits a quasi-uniformity. The problem whether every topological space admits a complete quasi-uniformity is considered in [**3**, Problem C], where an example is given of a T_1-space that admits a complete, but no convergence complete, quasi-uniformity. In this note we show that a locally compact separable normal M-space of D. K. Burke and E. K. van Douwen admits no complete quasi-uniformity, thereby answering an old question in the theory of quasi-uniform spaces. Moreover, we show that this space is an almost finitely-fully normal countably paracompact space that is not almost $\aleph_0$-fully normal. It is interesting to compare these results with the recent results of K. P. Hart [**4**] that M. E. Rudin's Dowker space is both orthocompact and finitely-fully normal; for it follows readily from Hart's results that, while Rudin's space is not almost $\aleph_0$-fully normal, it does admit a complete quasi-uniformity.

1. Definitions and a lemma. A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ such that (a) each member of $\mathcal{U}$ is a reflexive relation on X, and (b) if $U \in \mathcal{U}$ then $V \circ V \subset U$ for some $V \in \mathcal{U}$. The pair $(X, \mathcal{U})$ is called a *quasi-uniform space*. A filter $\mathcal{F}$ on $(X, \mathcal{U})$ is a *Cauchy filter* provided that for each $U \in \mathcal{U}$ there exists $p \in X$ so that $U(p) \in \mathcal{F}$, and $(X, \mathcal{U})$ is said to be *complete* provided that every Cauchy filter has a cluster point. The topology $\tau(\mathcal{U}) = \{G \subset X\colon$ for each $x \in G$ there is $U \in \mathcal{U}$ with $U(x) \subset G\}$ is called the *topology induced* by $\mathcal{U}$. A topological space (X, τ) *admits* $\mathcal{U}$ provided that τ is the topology induced by $\mathcal{U}$. Let (X, τ) be a topological space and let $\mathcal{B}$ be the collection of reflexive transitive relations V on X for which $V(x) \in \tau$ for all $x \in X$. Then $\mathcal{B}$ is a filterbase for a quasi-uniformity $\mathcal{U}$. Moreover, using the observation of W. J. Pervin [**8**] that for each open set G, $G \times G \cup (X \setminus G) \times X \in \mathcal{U}$, we see that (X, τ) admits $\mathcal{U}$. It is known that this quasi-uniformity $\mathcal{U}$ is complete if and only if every ultrafilter on X without a cluster point has a closure-preserving subcollection without a cluster point [**3**, p. 59]. Consequently, a

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regular space that is either almost real-compact or weakly orthocompact admits a complete quasi-uniformity.

Throughout, if $\mathcal{R}$ is a cover of a space X, and A is a subset of some member of $\mathcal{R}$, we say that A is a *refiner* of $\mathcal{R}$. A space X is *almost finitely-fully normal* (*almost* $\aleph_0$*-fully normal*) [6] provided that if $\mathcal{C}$ is an open cover of X there is an open refinement $\mathcal{R}$ of $\mathcal{C}$ with the property that if M is a finite (countable) set and M is a refiner of $\{\mathrm{st}(x, \mathcal{R}) \mid x \in X\}$, then M is also a refiner of $\mathcal{C}$.

We begin with a slight extension of a result of G. Aquaro [1] and K. Morita [7, Lemma 4.3].

LEMMA 1. *A space X is almost finitely-fully normal (almost $\aleph_0$-fully normal) if and only if X is normal, and for each open cover $\mathcal{C}$ of X there is a locally finite open cover $\mathcal{R}$ of X so that if M is a refiner of $\mathcal{R}$ and M is a finite (countable) set, then M is a refiner of $\mathcal{C}$.*

2. The example. The example X under consideration is described completely in [2]. For our purposes it is enough to know the following details: The ground set is $\mu \cup (\omega \times \omega)$, μ and ω considered disjoint, where μ is a regular cardinal, and there is a collection $F = \{f_\alpha \colon \alpha \in \mu\}$ such that

(0) for each $\alpha \in \mu$, $f_\alpha \in {}^\omega\omega$,

(1) each f_α is nondecreasing,

(2) $f_\alpha <^* f_\beta$ if $\alpha < \beta$,

(3) there is no $g \in {}^\omega\omega$ so that $f_\alpha \leqslant^* g$ for all $\alpha \in \mu$.

(As usual, we say $f <^* g$ provided that for all but finitely many $n \in \omega$, $f(n) < g(n)$.)

The points of $\omega \times \omega$ are isolated, and basic open sets about $\alpha \in \mu$ with $0 \leqslant \beta < \alpha < \mu$ and $m \in \omega$ are of the form

$$U(\alpha, \beta, m) = (\beta, \alpha] \cup \{\langle k, n\rangle \colon k \geqslant m \text{ and } f_\beta(k) < n \leqslant f_\alpha(k)\}.$$

If $\alpha \in \mu$ and $\alpha = 0$, basic open sets about α are of the form

$$U(0, \beta, m) = \{0\} \cup \{\langle k, n\rangle \colon k \geqslant m, n \leqslant f_0(k)\} \quad \text{where } m \in \omega.$$

LEMMA 2. *If S is a cofinal subset of μ, then $A = \{k \in \omega \colon \langle f_s(k)\rangle_{s \in S}$ is eventually bounded$\}$ is an initial segment of ω.*

PROOF. Since each f_α is nondecreasing, we note that if $a < b < \omega$ and $b \in A$, then $a \in A$. Assume $A = \omega$. Then for all $n \in \omega$ there are $s_n \in S$ and $k_n \in \omega$ such that if $s \in S$ and $s > s_n$ then $f_s(n) < k_n$. Define $g \colon \omega \to \omega$ by $g(n) = k_n$. There is a tail S' of S such that if $s \in S'$ then $f_s(n) < g(n)$ for all $n \in \omega$. Let $f_\alpha \in F$. There is an $s \in S'$ with $\alpha < s$; by (2), $f_\alpha \leqslant^* f_s$, and thus $f_\alpha \leqslant g$. We have shown that (3) fails—a contradiction. ■

X admits no complete quasi-uniformity. For each $x \in \mu$ and $m \in \omega$, set $F(x, m) = \{\langle k, n\rangle \colon k \geqslant m \text{ and } f_x(k) < n\}$ and let $\mathcal{F}$ be the filter for which $\{F(x, m) \colon x \in \mu$ and $m \in \omega\}$ is a filter base. Clearly no point of $\omega \times \omega$ is a cluster point of $\mathcal{F}$. If $\rho \in \mu$ and $m \in \omega$, then $U(\rho, 0, 0) \cap F(\rho, m) = \varnothing$. Therefore $\mathcal{F}$ is a filter without a cluster point. We show that $\mathcal{F}$ is a Cauchy filter with respect to each quasi-uniformity that X admits. Let $\mathcal{V}$ be such a quasi-uniformity, let $V \in \mathcal{V}$ and let $W \in \mathcal{V}$ so $W^2 \subset V$. For each $x \in \mu$ choose $\beta_x \in \mu$ and $m_x \in \omega$ so that $U(x, \beta_x, m_x) \subset W(x)$. By the Pressing-Down Lemma, there is a cofinal subset S of μ, $\beta \in \mu$ and $j \in \omega$ so that, for all $s \in S$, $U(s, \beta, j) \subset W(s)$. We note that $\beta < s$ for each $s \in S$. Set $A' = \{k \in \omega$: $\langle f_s(k) \rangle_{s \in S}$ is bounded$\}$. By Lemma 2, A' is finite. There is $e \in \omega$ so that if $k \geqslant e$ then $\{f_s(k)$: $s \in S\}$ is unbounded. Let $r = \max\{e, j\}$. For each $k \geqslant r$, define a function a_k: $\omega \to S$ by letting $a_k(n)$ be the least ordinal $\alpha \in S$ so $f_\alpha(k) \geqslant n$. Let $\gamma = \sup\{a_k(n)$: $k \geqslant r, n \in \omega\}$; there exists $s_0 \in S$ with $\gamma < s_0 < \mu$. Let $k \geqslant r$ and let $m \in \omega$. Then $a_k(m) \in U(s_0, \beta, j) \subset W(s_0)$ so $W(a_k(m)) \subset W^2(s_0) \subset V(s_0)$. We show that $F(\beta, r) \subset V(s_0)$ so $V(s_0) \in \mathfrak{F}$ as required. Let $\langle k, n \rangle \in F(\beta, r)$. Then $k \geqslant r \geqslant j$ and $f_\beta(k) < n$ so

$$\langle k, n \rangle \in U(a_k(n), \beta, j) \subset W(a_k(n)) \subset V(s_0). \quad \blacksquare$$

COROLLARY. *The space X is not weakly orthocompact.* (*Indeed, X is a transitive space that is not preorthocompact*; *see* [**3**, *Lemma* 6.16 *and Corollary* 5.11].)

For the nonce, mimicking terminology of E. Hewitt, we say a topological space is q-complete provided it admits a complete quasi-uniformity. In this terminology, Proposition 3.12 of [**3**] shows that the perfect preimage of a q-complete space is q-complete. The present example shows that this result does not obtain if perfect maps are replaced by quasi-perfect maps.

3. Further properties of the example. *X is almost finitely-fully normal.* Let $\mathfrak{B}$ be an open cover of X. Without loss of generality, we assume that for each $x \in \mu$ there is $B_x \in \mathfrak{B}$ of the form $B_x = U(x, \beta_x, m_x)$ and that all remaining members of $\mathfrak{B}$ are isolated points. Since μ is a regular cardinal, by the Pressing-Down Lemma, there are a cofinal subset S of μ, $\beta \in \mu$, and $m \in \omega$ so that $\beta_x = \beta$ and $m_x = m$ for all $x \in S$.

Set $A = \{k \in \omega$: $\langle f_s(k) \rangle_{s \in S}$ is eventually bounded$\}$. By Lemma 2, A is a finite set. Let h be a natural number exceeding $\max(A)$. Set $R = (\beta, \mu) \cup \{\langle k, n \rangle$: $k \geqslant h$, m and $f_\beta(k) < n\}$. Then R is an open set, and since $\mu \setminus R$ is compact there is a finite subset $\mathfrak{B}'$ of $\mathfrak{B}$ so that $\mu \setminus R \subset \bigcup \mathfrak{B}'$. Let

$$\mathfrak{R} = \mathfrak{B}' \cup \{R\} \cup \{\{x\} : x \notin \bigcup (\mathfrak{B}' \cup \{R\})\}.$$

Then $\mathfrak{R}$ is a locally finite open cover of X. Let M be a finite refiner of $\mathfrak{R}$. By Lemma 1 it suffices to show that M is a refiner of $\mathfrak{B}$. We assume $M \subset R$. Set $M_1 = M \cap \mu$ and set $M_2 = M \cap (\omega \times \omega)$. List the members of M_2 as $\langle k_1, n_1 \rangle$, $\langle k_2, n_2 \rangle, \ldots$, $\langle k_j, n_j \rangle$, where $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_j$. Let $q = \max\{n_1, n_2, \ldots, n_j\}$. There is an $s' \in S$ so that $s' > \max M_1$. Since $k_1 \geqslant h$, $k_1 \notin A$ so there is an $s \in S$ with $s > s'$ so $f_s(k_1) > q$. Let $1 \leqslant d \leqslant j$. Since $M \subset R$, $f_\beta(k_d) < n_d \leqslant q < f_s(k_1) \leqslant f_s(k_d)$. Thus $M_2 \subset \{\langle k, n \rangle$: $k \geqslant m$, $f_\beta(k) < n \leqslant f_s(k)\} \subset U(s, \beta, m)$. It follows that $M \subset U(s, \beta, m)$.

X is not almost $\aleph_0$-fully normal. Set $\mathfrak{B}' = \{U(x, 0, 0)$: $x \in \mu\}$ and set $\mathfrak{B} = \mathfrak{B}' \cup \{\{x\}$: $x \in \omega \times \omega\}$. Suppose $\mathfrak{L}$ is an open locally finite cover of X with the property that every countable refiner of $\mathfrak{L}$ is a refiner of $\mathfrak{B}$. Since $\mathfrak{L} | \mu$ is locally finite, and hence finite, only finitely many members of $\mathfrak{L}$ meet μ; list these members as L_1, $L_2, \ldots, L_z$. For each $x \in \mu$, there are k_x, β_x and n_x so that $x \in U(x, \beta_x, n_x) \subset L_{k_x}$,

where $U(x, \beta_x, n_x)$ is a basic open set about x. By the Pressing-Down Lemma, there is a cofinal subset $\tilde{S}$ of μ, $\tilde{m} \in \omega$, $\tilde{k}$ with $1 \leqslant \tilde{k} \leqslant z$, and $\tilde{\beta} \in \mu$ so that for each $x \in \tilde{S}$, $k_x = \tilde{k}$, $\beta_x = \tilde{\beta}$ and $n_x = \tilde{m}$. Set $\tilde{A} = \{k \in \omega\colon \langle f_s(k) \rangle_{s \in \tilde{S}}$ is eventually bounded$\}$. Then $\tilde{A}$ is finite and there is a natural number $\tilde{h}$ exceeding $\max(\tilde{A})$. Set

$$\tilde{R} = (\tilde{\beta}, \mu) \cup \left\{\langle k, n \rangle : k \geqslant \tilde{h}, \tilde{m} \text{ and } f_{\tilde{\beta}}(k) < n\right\}.$$

Then $\tilde{R} \subset L_{\tilde{k}}$ so each countable subset of $\tilde{R}$ is a refiner of $\mathfrak{B}$. Let $D = \tilde{R} \setminus \mu$. Then D is a refiner of $\mathfrak{B}$—a contradiction. ■

In his thesis [**5**, Theorem 2.2.10], H. J. K. Junnila proves that a space X is 2-fully normal if and only if it is almost 2-fully normal and for each open cover $\mathcal{C}$ of X there is a reflexive relation V on X such that, for each $x \in X$, $V(x)$ is open and such that, for each $x \in X$ and $y \in V(x)$, $V(x) \cup V(y)$ is a refiner of $\mathcal{C}$. In particular, every orthocompact almost 2-fully normal space is 2-fully normal. It is unknown whether the converse holds; the referee suggests that the space we have considered above could possibly be used in the construction of a counterexample.

Since the space X was constructed as an example of a space that does not have a countably-compact-ification, it is interesting to note that nearly the same method of proof establishes that the following countably compact normal space Y is not q-complete.

Let $\langle A_\alpha \rangle_{\alpha \in \mu}$ be an increasing maximal tower on ω, where μ is a regular cardinal. Let $Y = \mu \cup \omega$ and, as usual, define a topology on Y by specifying the following neighborhoods: Points of ω are isolated. If $0 \leqslant \beta < \alpha < \mu$ and F is a finite subset of ω, set

$$U(\alpha, \beta, F) = (\beta, \alpha] \cup \left[A_\alpha \setminus A_\beta\right] \setminus F,$$

and if $\alpha = 0$ and F is a finite subset of ω, set

$$U(0, \beta, F) = \{0\} \cup (A_0 \setminus F).$$

Then $\mathfrak{F} = \text{fil}\{(\omega \setminus A_\alpha) \setminus F\colon \alpha \in \mu$ and F is a finite subset of $\omega\}$ is a filter without a cluster point that is a Cauchy filter with respect to each quasi-uniformity that Y admits.

The similarity of the methods of proof that X and Y are not q-complete is not just a coincidence. The basic neighborhoods of points of μ in X can be defined in terms of the following tower on $\omega \times \omega$: $\langle \{\langle k, n \rangle\colon k \in \omega, n \leqslant f_\alpha(k)\} \rangle_{\alpha \in \mu}$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITÄT BERN, CH-3012 BERN, SWITZERLAND

DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECH INSTITUTE & STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061 (Current address of both authors)