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A REMARK ON REFINABLE MAPS AND CALMNESS

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ABSTRACT. It is shown that if $r: X \to Y$ is a refinable map between compacta and Y is calm, then r is a shape equivalence. As a corollary, if $r: X \to Y$ is a refinable map between compacta and either X or Y is S^n-like ($n \geq 1$), then r is a shape equivalence, where S^n denotes the n-sphere.

0. Introduction. In [**9**], J. Ford and J. W. Rogers, Jr. introduced the notion of refinable maps and they proved several results about these maps. In [**10**], we showed that every refinable map does not preserve shape (cf. [**12**]) and introduced the notion of pseudo-isomorphisms in pro-category, and by using this notion we proved that if $r: X \to Y$ is a refinable map between compacta and Y is an FANR, then r is a shape equivalence. In this paper, we will give the following more general result: if $r: X \to Y$ is a refinable map between compacta and Y is calm, then r is a shape equivalence. The notion of calmness was introduced by Z. Čerin [**4**]. It is well known that the dyadic solenoid is calm but not movable [**4**]. As a corollary, we obtain that if $r: X \to Y$ is a refinable map between compacta and either X or Y is S^n-like ($n \geq 1$), then r is a shape equivalence. In relation to the above result, the following are known; if $r: X \to Y$ is a refinable map between compacta, then r is a weakly confluent map [**9**], moreover;

(1) if Y has property $[K]$ due to J. L. Kelley [**15**, (3.2)], then r is a confluent map [**14**],

(2) if Y is locally connected, then r is a monotone map [**9**],

(3) if Y is locally n-connected ($n \geq 1$), then r is a UV^n-map [**11**], and

(4) if Y is an ANR (locally contractible), then r is a cell-like map [**11**].

Several properties concerning the notions of refinable maps, ARI-maps, AI-maps, calmness, AANR and quasi-ANR etc. have been studied in [**1, 3–14, 16, 17** etc.].

1. Definitions and notations. For a metric space X, if x and y are points of X, $d(x,y)$ denotes the distance from x to y. A map $r: X \to Y$ between compacta is *refinable* [**9**] if for any $\epsilon > 0$ there is a surjective map $f: X \to Y$ such that $\operatorname{diam} f^{-1}(y) < \epsilon$ for each $y \in Y$ and $d(r,f) = \sup\{d(r(x), f(x)) \mid x \in X\} < \epsilon$. Note that every refinable map is surjective, every near-homeomorphism is refinable and if there is a refinable map from a compactum X to a compactum Y, then X is Y-like. But simple examples show that the converse of any of these assertions is not true. A compactum X is *calm* [**4**] if whenever $X \subset M \in$ ANR, there is a neighborhood V of X in M such that for any neighborhood U of X in M there is a neighborhood W of X in M, $W \subset U$, such that if $f, g: Y \to W$ are maps with $f \simeq g$ in V, then $f \simeq g$ in U for all $Y \in$ ANR. Let K be an arbitrary category. By **K**, we mean the category of inverse systems in K and system maps in K, also by pro-K, the

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homotopy category of $\mathbf{K}$. An inverse system $\{X_\alpha, p_{\alpha\alpha}, A\}$ of pro-K is *calm* if there is $\alpha_0 \in A$ such that

$$(*)\qquad \begin{array}{l} \text{for any } \alpha \in A\,(\alpha \geq \alpha_0) \text{ there is } \beta \in A,\ \beta \geq \alpha, \text{ such that} \\ \text{if } f, g\colon Y \to X_\beta \text{ are morphisms in } K \text{ with } p_{\alpha_0\beta} f = p_{\alpha_0\beta} g, \\ \text{then } p_{\alpha\beta} f = p_{\alpha\beta} g \text{ for all } Y \in K. \end{array}$$

It is easily seen that a compactum X is calm iff there is an inverse system $\{X_n, [p_{nn+1}]\}$ in pro-HCW which is calm and associated with X, where HCW is the category of spaces having the homotopy type of CW-complexes and homotopy classes of maps. A system map $\mathbf{f} = \{f, f_\beta, B\}\colon \{X_\alpha, p_{\alpha\alpha'}, A\} \to \{Y_\beta, q_{\beta\beta'}, B\}$ of $\mathbf{K}$ is a *pseudo-isomorphism* [10] if for each $\beta \in B$ and each $\alpha \geq f(\beta)$ there exist $g(\alpha,\beta) \geq \beta$ and a morphism $g_{(\alpha,\beta)}\colon Y_{g(\alpha,\beta)} \to X_\alpha$ such that

(**) for every $\beta' \geq g(\alpha,\beta)$ there exist $h(\beta') \geq \alpha$ and a morphism $h_{\beta'}\colon X_{h(\beta')} \to Y_{\beta'}$ such that

$$f_\beta p_{f(\beta)\alpha} g_{(\alpha,\beta)} = g_{\beta g(\alpha,\beta)} \quad \text{and} \quad g_{(\alpha,\beta)} q_{g(\alpha,\beta)\beta'} h_{\beta'} = p_{\alpha h(\beta')}.$$

A morphism $f\colon \mathbf{X} = \{X_\alpha, p_{\alpha\alpha'}, A\} \to \mathbf{Y} = \{Y_\beta, q_{\beta\beta'}, B\}$ of pro-K is a pseudo-isomorphism if it has a pseudo-isomorphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ of $\mathbf{K}$ as the representation, i.e. $f = [\mathbf{f}]$.

THEOREM (1.1) [7]. *A compactum X is an FANR iff X is calm and movable.*

THEOREM (1.2) [1]. *A compactum X is an $AANR_N$ iff X is an $AANR_C$ and an FANR.*

2. Calmness and pseudo-isomorphisms. In this section, we show that if $r\colon X \to Y$ is a refinable map between compacta and Y is calm, then r is a shape equivalence. First, we prove the following lemma (cf. [10, (2.1)].

LEMMA (2.1). *For a category K, if $f\colon \mathbf{X} = \{X_\alpha, p_{\alpha\alpha'}, A\} \to \mathbf{Y} = \{Y_\beta, q_{\beta\beta'}, B\}$ is a pseudo-isomorphism in pro-K and $\mathbf{Y}$ is calm, then f is an isomorphism in pro-K.*

PROOF. Let $\mathbf{f} = \{f, f_\beta, B\}\colon \mathbf{X} \to \mathbf{Y}$ be a pseudo-isomorphism of $\mathbf{K}$ such that $f = [\mathbf{f}]$. Since $\mathbf{Y}$ is calm, there is $\beta_0 \in B$ satisfying the condition (*). Let $\beta \geq \beta_0$. Since $\mathbf{f}$ is a pseudo-isomorphism, for each $\alpha \geq f(\beta)$ there exist $g(\alpha,\beta) \geq \beta$ and a morphism $g_{(\alpha,\beta)}\colon Y_{g(\alpha,\beta)} \to X_\alpha$ such that

$$(1)\qquad f_\beta p_{f(\beta)\alpha} g_{(\alpha,\beta)} = q_{\beta g(\alpha,\beta)}$$

and the condition (**) is satisfied. By the choice of β_0, we can choose $\beta' \geq g(\alpha,\beta)$ such that if $f, g\colon Y \to Y_{\beta'}$ are morphisms in K with $q_{\beta_0\beta'} f = q_{\beta_0\beta'} g$, then $q_{g(\alpha,\beta)\beta'} f = q_{g(\alpha,\beta)\beta'} g$ for all $Y \in K$. Also, by the condition (**), there exist $h(\beta') \geq f(\beta')$ and a morphism $h_{\beta'}\colon X_{h(\beta')} \to Y_{\beta'}$ such that

$$(2)\qquad g_{(\alpha,\beta)} q_{g(\alpha,\beta)\beta'} h_{\beta'} = p_{\alpha h(\beta')}$$

and

$$(3)\qquad f_\beta p_{f(\beta)h(\beta')} = q_{\beta\beta'} f_{\beta'} p_{f(\beta')h(\beta')}.$$

Then by (1)–(3), we have

$$
\begin{aligned}
q_{\beta_0 g(\alpha,\beta)}(q_{g(\alpha,\beta)\beta'}h_{\beta'}) &= q_{\beta_0\beta}f_\beta p_{f(\beta)\alpha}g_{(\alpha,\beta)}q_{g(\alpha,\beta)\beta'}h_{\beta'} \\
&= g_{\beta_0\beta}f_\beta p_{f(\beta)\alpha}p_{\alpha h(\beta')} \\
&= q_{\beta_0\beta}f_\beta p_{f(\beta)h(\beta')} \\
&= q_{\beta_0\beta}q_{\beta\beta'}f_{\beta'}p_{f(\beta')h(\beta')} \\
&= q_{\beta_0 g(\alpha,\beta)}(q_{g(\alpha,\beta)\beta'}f_{\beta'}p_{f(\beta')h(\beta')}).
\end{aligned}
\tag{4}
$$

By the condition $(*)$, we have

$$q_{g(\alpha,\beta)\beta'}h_{\beta'} = q_{g(\alpha,\beta)\beta'}f_{\beta'}p_{f(\beta')h(\beta')}. \tag{5}$$

Hence by (2) and (5), we have

$$g_{(\alpha,\beta)}q_{g(\alpha,\beta)\beta'}f_{\beta'}p_{f(\beta')h(\beta')} = g_{(\alpha,\beta)}q_{g(\alpha,\beta)\beta'}h_{\beta'} = p_{\alpha h(\beta')}. \tag{6}$$

By (1) and (6), we can conclude that the morphism $f\colon \mathbf{X} \to \mathbf{Y}$ is an isomorphism in pro-K.

THEOREM (2.2) [**10**, 1.5 THEOREM]. *If $r\colon X \to Y$ is a refinable map between compacta, then r induces a pseudo-isomorphism in pro-HCW.*

By using (2.1) and (2.2), we obtain the following theorem (cf. [**3**, 3.3 Corollary]).

THEOREM (2.3). *If $r\colon X \to Y$ is a refinable map between compacta and Y is calm, then r is a shape equivalence, i.e.,* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

Combining Theorems (1.1), (1.2) and (2.3), we have

COROLLARY (2.4) [**10**, 2.2 THEOREM]. *If $r\colon X \to Y$ is a refinable map between compacta and Y is an FANR, then r is a shape equivalence.*

COROLLARY (2.5). *If $r\colon X \to Y$ is a refinable map between compacta and Y is an $AANR_N$, then r is a shape equivalence.*

REMARK (2.6). In the statement of (2.3), we cannot replace "calm" by "movable", also in (2.5), we cannot replace AANR_N by AANR_C (see [**10**, Examples (2.5), (2.6) and (2.8)]).

Let $\mathfrak{P}$ be a class of ANR-sets. A compactum X is $\mathfrak{P}$-*calm* [**4**] if whenever $X \subset M \in \mathrm{ANR}$, there is a neighborhood V of X in M such that for every neighborhood U of X in M there is a neighborhood W, $W \subset U$, such that if $f, g\colon Y \to W$ with $f \simeq g$ in V, then $f \simeq g$ in U for all $Y \in \mathfrak{P}$. By using this notion, we obtain a more general result than (2.3) as follows.

THEOREM (2.7). *If $r\colon X \to Y$ is a refinable map between compacta and X is $\mathfrak{P}$-like and Y is $\mathfrak{P}$-calm, then r is a shape equivalence.*

PROOF. Observe the proofs of [**10**, 1.5 Theorem] and (2.3).

COROLLARY (2.8). *If $r\colon X \to Y$ is a refinable map between compacta and if either X or Y is S^n-like $(n \geq 1)$, then r is a shape equivalence, where S^n denotes the n-sphere.*

PROOF. By [**9**, Corollary 3.1], we conclude that X and Y are S^n-like. Let $\{Y_i, p_{ii+1}\}$ be an inverse sequence such that each Y_i is the n-sphere ($= S^n$) and $Y = \operatorname{inv}\lim\{Y_i, p_{ii+1}\}$. Note that $\pi_n(p_{ii+1})\colon \pi_n(Y_{i+1}) \to \pi_n(Y_i)$ is either a monomorphism or a zero homomorphism for each $i = 1, 2, \ldots$. Hence we can conclude that Y is S^n-calm. Theorem (2.8) implies that r is a shape equivalence.

We finish this paper with some open questions.

Question 1. Does every refinable map preserve calmness?

Question 2. Does every refinable map preserve FANR? (For the partial positive answer, see [**10**, Theorems 2.4, 2.12].)

Question 3. Does every refinable map preserve AANR_N?
The positive answer of Question 1 implies the positive answer of Question 2 and the positive answer of Question 2 implies the positive answer of Question 3.

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