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UNIFORM ORDER-CONVERGENCE FOR COMPLETE LATTICES

GABRIELE H. GRECO

ABSTRACT. We introduce a purely lattice-theoretical definition of uniform order-convergence of a net of functions with values in a complete lattice. We will show that for completely distributive lattices the uniform order-convergence is induced by a uniformity.

A net $\{x_j\}_j$ of elements of a complete lattice L *order-converges to* a when $\operatorname{Lim\,inf}\{x_j\} = \operatorname{Lim\,sup}\{x_j\} = a$ [**1**]. We say that a net $\{f_j\}_j$ of functions on a set X to a complete lattice L *uniformly order-converges* to f if for every subset A of X:

(1) $\bigwedge f(A) \leqslant \operatorname{Lim\,inf}\{\bigwedge f_j(A)\}$,

(2) $\bigvee f(A) \geqslant \operatorname{Lim\,sup}\{\bigvee f_j(A)\}$.

The *interval topology* on a given complete lattice L is that defined by taking the closed intervals of L as subbase for the closed sets. Any complete lattice is compact for the interval topology [**2**]. It is well known that usually the interval topology is not Hausdorff and the order-convergence is not topological. However any direct product of complete chains is Hausdorff for the interval topology and its convergence is the order-convergence [**2**]. A completely distributive complete lattice is isomorphic to a closed sublattice of a direct product of complete chains (Raney's Theorem [**3, 1**]). Hence for a completely distributive complete lattice L the inteval topology is the uniform topology of a compact Hausdorff uniformity $\mathcal{U}_L$ and its convergence is the order-convergence.

THEOREM. *A net $\{f_j\}_j$ of functions on a set X to a completely distributive complete lattice L uniformly order-converges to f if and only if it uniformly converges to f relatively to $\mathcal{U}_L$.*

PROOF. A net of functions with values in a direct product of complete chains is uniformly (order-)convergent if and only if its projections in the coordinate spaces are uniformly (order-)convergent. Hence by the above quoted Raney Theorem it is enough to prove the Theorem when L is a complete chain C.

Let C be a chain with universal bounds 0, 1 and $\mathcal{U}_C$ its uniformity. Suppose $\{f_j\}$ is uniformly order-convergent to f: we will show that it is uniformly convergent to f relative to $\mathcal{U}_C$. Let V be a member of $\mathcal{U}_C$. By the compactness of $\mathcal{U}_C$ there is a finite open cover $\{V_i\}_i$ of C such that $V \supset \bigcup_i(V_i \times V_i)$. Since every finite open cover of the chain C has a finite refinement consisting of closed intervals, there exists a finite family $\{I_k\}_{k=1}^n$ of closed intervals such that for every k, $I_k \subset V_{i(k)}$ for some $i(k)$ and

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$\bigcup_{k=1}^{n} I_k = C$. Setting $A_k = \{x \in X: f(x) \in I_k\}$, we have $[\bigwedge f(A_k), \bigvee f(A_k)] \subset V_{i(k)}$ for $1 \leqslant k \leqslant n$. Since $V_{i(k)}$ is open, by the uniform order-convergence of $\{f_j\}$ to f there exists j_k such that for every $j \geqslant j_k$, $[\bigwedge f_j(A_k), \bigvee f_j(A_k)] \subset V_{i(k)}$. Choose $j_0 \geqslant j_k$ for every $1 \leqslant k \leqslant n$. We obtain for every k $[\bigwedge f(A_k), \bigvee f(A_k)] \times [\bigwedge f_j(A_k), \bigvee f_j(A_k)] \subset V_{i(k)} \times V_{i(k)} \subset V$. Hence by $\bigcup_{k=1}^{n} A_k = X$ the net $\{f_j\}$ uniformly converges to f.

Now suppose $\{f_j\}$ is uniformly convergent to f with respect to uniformity $\mathfrak{U}_C$. We will verify property (1) (dually the proof of (2) is obtained). Let A be a subset of X. If $[\bigwedge f(A), 1]$ is open, there is j_0 such that $\{f_j(x): j \geqslant j_0, x \in A\} \subset [\bigwedge f(A), 1]$, since the set $[0, \bigwedge f(A))^2 \cup [\bigwedge f(A), 1]^2$ is a member of $\mathfrak{U}_C$ and the net $\{f_j\}$ uniformly converges to f. Hence property (1) follows. Otherwise if $[\bigwedge f(A), 1]$ is not open, $\bigwedge f(A) = \bigvee\{b \in C: b < \bigwedge f(A)\}$. Let $b < \bigwedge f(A)$. The set $[0, \bigwedge f(A))^2 \cup (b, 1]^2$ is a member of $\mathfrak{U}_C$. Hence by the uniform convergence of $\{f_j\}$ to f, there is j_0 such that $\{f_j(x): x \in A, j \geqslant j_0\} \subset (b, 1]$. Then $\operatorname{Lim\,inf}\{\bigwedge f_j(A)\} \geqslant b$ for every $b < \bigwedge f(A)$. This implies property (1). $\square$

As a simple consequence of the above Theorem, we obtain a purely ordered characterization of the notion of equicontinuity.

Let L be a complete lattice and X a topological space. A function f on X to L is *lower* (*upper*) *semicontinuous* at x_0 if $\operatorname{Lim\,inf}_{x \to x_0} f(x) \geqslant f(x_0)$ ($\operatorname{Lim\,sup}_{x \to x_0} f(x) \leqslant f(x_0)$). A family $\mathbf{F}$ of functions on X to L is *lower* (*upper*) *equi-semicontinuous* if for every subfamily $\mathbf{A} \subset \mathbf{F}$ the function $\bigwedge \mathbf{A}$ ($\bigvee \mathbf{A}$) is lower (upper) semicontinuous. This definition of lower (upper) equi-semicontinuity agrees with the usual one for a closed interval of the real field completed by adjoining universal bounds $-\infty$ and $+\infty$.

COROLLARY. *Let L be a completely distributive complete lattice and $\mathfrak{U}_L$ its uniformity. A family $\mathbf{F}$ is equi-continuous at x_0 with respect to $\mathfrak{U}_L$ if and only if $\mathbf{F}$ is both lower and upper equi-semicontinuous at x_0.*

In view of the Raney Theorem, our Theorem entails

GENERALIZED DINI THEOREM. *Let X be a topological compact space and let L be a completely distributive complete lattice. If a net $\{f_j\}_j$ of continuous functions on X valued in L increasingly pointwise converges to a continuous function f, then $\{f_j\}_j$ uniformly order-converges to f.*

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO, 38050 POVO (TN), ITALY