HIGHER WHITEHEAD GROUPS OF CERTAIN BUNDLES
OVER SEIFERT MANIFOLDS

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Abstract. Vanishing results for \( \text{Wh}_j(\pi_1 M) \otimes R \) (\( R = \mathbb{Z}, \mathbb{Q}, \) or \( \mathbb{Z}[1/2] \)) are obtained when \( M \) is a closed aspherical manifold which is the total space of a bundle over an insufficiently large Seifert manifold with infinite fundamental group of hyperbolic type. Allowable fibers include Riemannian flat manifolds and closed aspherical manifolds with poly-\( \mathbb{Z} \) fundamental groups. Corollaries concern the homotopy groups of the group \( \text{TOP}(M) \) of self-homeomorphisms of \( M \).

Let \( R \) be a subring of the rational numbers, \( n \) a nonnegative integer, and \( N \) a connected manifold.

HYPOTHESIS \( A(n, R) \). \( \mathbb{Z}^N \) is a right regular Noetherian ring and \( \text{Wh}_j(\pi_1 N) \otimes R = 0 \) for \( 0 \leq j \leq n \).

\( N \) satisfies Hypothesis \( A(\infty, R) \) if \( N \) satisfies Hypothesis \( A(n, R) \) for all \( n \). It is known that if \( \pi_1 N \) is a poly-\( \mathbb{Z} \) group then \( N \) satisfies \( A(\infty, \mathbb{Z}) \). If \( \pi_1 N \) is a Bieberbach group then \( N \) satisfies \( A(1, \mathbb{Z}), A(3, \mathbb{Z}[1/2]), \) and \( A(\infty, \mathbb{Q}) \) [FH1, N2, N3]. It is conjectured that if \( \pi_1 N \) is a Bieberbach group then \( N \) satisfies \( A(\infty, \mathbb{Z}) \).

Waldhausen's results on the \( K \)-theory of generalized free products [W], especially Corollaries 17.1.3 and 17.2.3, yield the following lemma. Recall that the \( j \)th Whitehead group of a group \( G \) is the \( j \)th homotopy group of a space \( \text{Wh}_{\mathbb{Z}}(G) \) and that these corollaries establish homotopy Cartesian squares involving these Whitehead spaces.

**lemma.** Let \( M \) be the total space of a fiber bundle over a compact, connected manifold \( K \) with fiber \( N \). Assume that \( N \) satisfies Hypothesis \( A(n, R) \). If \( K \) is a surface other than \( S^2 \) or \( \mathbb{R} P^2 \), of if \( K \) is a Haken 3-manifold, then for \( 0 \leq j \leq n \), \( \text{Wh}_j(\pi_1 M) \otimes R = 0 \).

**proof.** An \( N \)-bundle over a surface other than \( S^2 \) or \( \mathbb{R} P^2 \) is built up from copies of \( N \times D^2 \) by amalgamating along copies of \( N \times D^1 \) or \( N \)-bundles over \( S^1 \), while an \( N \)-bundle over a Haken 3-manifold is built up from copies of \( N \times D^3 \) by amalgamating along \( N \)-bundles over incompressible surfaces (which are not \( S^2 \) or \( \mathbb{R} P^2 \)). The arguments for the two cases are essentially the same.

When the base of the bundle is a surface, the integral group rings for the amalgamating subgroups are \( \mathbb{Z} \pi_1 N \) or twisted Laurent extensions of \( \mathbb{Z} \pi_1 N \); in either
case the group ring is right regular Noetherian (and hence coherent). This implies, by Corollary 4.2 of \[W\], that \( \mathbb{Z} \pi_1 M \) is right regular coherent when \( K \) is a surface, so Corollaries 17.1.3 and 17.2.3 of \[W\] are applicable when \( K \) is a surface or Haken 3-manifold.

These corollaries give homotopy Cartesian squares with the Whitehead space of a free product with amalgamations or an HNN extension as the lower right-hand corner of the square. Our hypothesis is that, after tensoring with \( R \), the other three spaces in the square are \( n \)-connected. This implies that the lower-hand space in the square is also \( n \)-connected after tensoring with \( R \). (To verify \( 0 \)-connectedness, use the argument on p. 250 of \[W\] based on the Bass-Heller-Swan inclusion \( Wh_0(G) \rightleftharpoons Wh_1(G \times \mathbb{Z}) \).) Repeated applications of this \( n \)-connectedness observation complete the proof of the Lemma.

**Proposition.** Let \( \pi \) be a group and \( f: \pi \to G \) an epimorphism onto a finite group, \( R \) a subring of the rational numbers, and \( n \) a nonnegative integer. Suppose that for every hyperelementary subgroup \( H \) of \( G \) the higher Whitehead groups of \( f^{-1}(H) \) satisfy \( Wh_j(f^{-1}(H)) \otimes R = 0 \) for \( 0 \leq j \leq n \). Then \( Wh_j(\pi) \otimes R = 0 \) for \( 0 \leq j \leq n \).

**Proof.** By \[W\] there is a long exact sequence for any group \( \Gamma \):

\[
(*) \quad Wh_{j+1}(\Gamma) \to h_j(B\Gamma; K\mathbb{Z}) \xrightarrow{l_j} K_j(\mathbb{Z}\Gamma) \to Wh_j(\Gamma) \to \cdots \to Wh_0(\Gamma) \to 0,
\]

where \( h_j(\; ; K\mathbb{Z}) \) is the generalized homology theory arising from the spectrum \( K\mathbb{Z} \) for algebraic \( K \)-theory and \( B\Gamma \) is the classifying space of \( \Gamma \). This sequence remains exact when tensored with \( R \). From (*) it follows that the conclusion of the theorem is equivalent to the statement; \( l_j \) is an isomorphism for all \( j \) such that \( 0 \leq j \leq n - 1 \) and an epimorphism for \( j = n \).

If \( H \) is a subgroup of \( G \) define

\[
m_j(H) = h_j(Bf^{-1}(H); K\mathbb{Z}) \otimes R, \quad k_j(H) = K_j(\mathbb{Z}f^{-1}(H)) \otimes R,
\]

and if \( H \) and \( K \) are subgroups of \( G \) and \( g \in G \) are such that \( gHg^{-1} \subset K \), let \( (H, g, K) \) be the homomorphism \( H \to K \) given by conjugation by \( g \).

Given \( I = (H, g, K) \), there is an induction map \( I_*: k_j(H) \to k_j(K) \) ("induced map") and a restriction map \( I^*: k_j(K) \to k_j(H) \) ("transfer"). There is also an induction map \( I_*: m_j(H) \to m_j(K) \) corresponding to the map in homology induced by \( Bf^{-1}(H) \to Bf^{-1}(K) \) and a restriction map \( I^*: m_j(K) \to m_j(H) \) corresponding to the homology transfer. According to [FH2], \( l_j: m_j(H) \to k_j(H) \) is natural with respect to induction and restriction, and \( k_j(\; ) \) is a Frobenius module over Swan's Frobenius functor \( G_0(\; ) \otimes R \), where \( G_0(H) \) is the Grothendieck group of integral representations of \( H \).

Let \( C \) be the collection of hyperelementary subgroups of \( G \). Define \( m_j(C) = \bigoplus_{H \in C} m_j(H) \) and \( k_j(C) = \bigoplus_{H \in C} k_j(H) \). Consider the following commutative diagrams:
Since $G_0(\ ) \otimes R$ satisfies hyperelementary induction [Sw], it follows from [Dr, Proposition 1.2] (also see [N, Theorem 6.2.7]) that $I^*: k_j(G) \to k_j(C)$ is injective and $I^*: k_j(C) \to k_j(G)$ is surjective. By [N, Lemma 6.2.8] $I^*: m_j(C) \to m_j(G)$ is surjective and $I^*: m_j(G) \to m_j(C)$ is injective. By hypothesis $l_j: m_j(C) \to k_j(C)$ is an isomorphism for $0 \leq j \leq n - 1$ and an epimorphism for $j = n$. A diagram chase reveals that $l_j: m_j(G) \to k_j(G)$ is an isomorphism for $0 \leq j \leq n - 1$ and an epimorphism for $j = n$, completing the proof of the Proposition.

Let $K^3$ be one of the insufficiently large Seifert manifolds with three exceptional orbits over $S^2$ given by the invariants $(b; (0,0,0,0); (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ in the notation of [O]. Let $\pi_1 K^3 \to \delta(\alpha_1, \alpha_2, \alpha_3)$ be the quotient by the image in the fundamental group of any regular fiber: here $\delta(\alpha_1, \alpha_2, \alpha_3)$ is the orientation-preserving subgroup of a triangle group.

**Main Theorem.** Let $M$ be the total space of a bundle with fiber $N$ and base $K^3$, where $K^3$ is one of the Seifert manifolds described above. If $\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} < 1$ and $N$ satisfies Hypothesis A($n$, $R$), then $\text{Wh}_j(\pi_1 M) \otimes R = 0$ for $0 \leq j \leq n$.

**Proof.** This argument is essentially that of [P] and relies on the fact that $Q = \delta(\alpha_1, \alpha_2, \alpha_3)$ is a hyperbolic triangle group if $\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} < 1$. In [P and S] it is shown that $Q$ has an epimorphism $h: Q \to G$ to a nonhyperelementary finite group $G$. Composition gives an epimorphism $f: \pi_1 M \to \pi_1 K \to Q \xrightarrow{h} G$.

Let $H$ be a subgroup of $G$. If the covering space of $M$ corresponding to $f^{-1}(H)$ is an $N$-bundle over a Haken manifold (i.e. if $h^{-1}(H)$ is not a triangle group), then the Lemma is applicable. As $H$ runs over the hyperelementary subgroups of $G$, though, some of the $h^{-1}(H)$'s may be triangle subgroups of $Q$, so this observation is not enough to finish the proof. However, hyperbolic triangle groups contain only finitely many triangle subgroups, so we may induce on the number $t(Q)$ of proper triangle subgroups in $Q$. If $t(Q) = 0$ then no $h^{-1}(H)$ is a triangle group and the Proposition and Lemma show that $\text{Wh}_j(\pi_1 M) \otimes R = 0$ for $0 \leq j \leq n$. If $t(Q) \geq 1$, then for any $h^{-1}(H)$ which is a triangle group, $t(h^{-1}(H)) < t(Q)$, so the Proposition and the inductive hypothesis imply $\text{Wh}_j(\pi_1 M) \otimes R = 0$ for $0 \leq j \leq n$.

Let $M$ and $N$ be as in the Main Theorem. Suppose $N$ satisfies the additional Hypothesis B. $N$ is a closed aspherical manifold and $S_{\text{Top}}(N \times I^j, \partial) = 0$ for $j + \dim(N) \geq 6$, where $S_{\text{Top}}(N \times I^j, \partial)$ is the structure set of topological surgery [KS].

An interesting class of manifolds which satisfy Hypothesis B is the closed aspherical manifolds with torsion-free poly- (finite or infinite cyclic) fundamental
group [FH3]. This class includes closed flat Riemannian manifolds and closed manifolds with poly-$\mathbb{Z}$ fundamental group. Suppose $N$ satisfies Hypothesis A($\infty$, $Q$) and B (for example, take $N$ to be a closed Riemannian flat manifold or a closed aspherical manifold with poly-$\mathbb{Z}$ fundamental group). By the Main Theorem, $\text{Wh}_j(\pi_1 M) \otimes Q = 0$ for all $j$, and by the main theorem of [S], $M$ will also satisfy Hypothesis B in many cases, including these:

(a) $\alpha_1$, $\alpha_2$ and $\alpha_3$ are all odd, or
(b) an odd prime $p$ divides one of the $\alpha$'s, say $\alpha_1$, and the group $Q(\alpha_1/p, \alpha_2, \alpha_3)$ is also a hyperbolic group of motions.

Let $\text{TOP}(M)$ be the topological group of self-homeomorphisms of $M$ and let $m = \dim(M)$. Suppose $M$ has the properties established for the examples considered above: $M$ satisfies Hypothesis B and all the Whitehead groups of $M$ vanish when tensored with the rationals. Theorem 4.5(B) of [FH2] now yields a computation of the rational homotopy groups $\pi_i(\text{TOP}(M)) \otimes Q$ for $1 \leq i \leq \phi_2(m)$, where $\phi_2(m)$ is the stable range for topological pseudoisotopy.

**Corollary 1.** For $1 \leq i \leq \phi_2(m)$,

$$
\pi_i(\text{TOP}(M)) \otimes Q = \begin{cases} 
\text{center}(\pi_1 M) \otimes Q, & i = 1, \\
\bigoplus_{j=1}^{\infty} H_{i+1,j-4}(M, Q), & i > 0, \ m \text{ odd}, \\
0, & i > 0, \ m \text{ even}.
\end{cases}
$$

**Remark.** The theorem of Farrell and Hsiang quoted above, while stated for the differentiable category in [FH2], is equally valid in the topological category. If $M$ is smoothable, Corollary 1 is true for the diffeomorphism group $\text{Diff}(M)$ in place of $\text{TOP}(M)$ provided $1 \leq i \leq \phi_1(m)$, where $\phi_1(m)$ is the stable range for smooth pseudoisotopy.

Now suppose $N$ is smoothable and satisfies Hypotheses A(1, $\mathbb{Z}$), A(3, $\mathbb{Z}[1/2]$), and B (again this will be the case if $N$ is a closed flat Riemannian manifold or if $N$ is a closed aspherical manifold with $\pi_1 N$ poly-$\mathbb{Z}$). By the Main Theorem and the main theorem of [S], $M$ satisfies Hypothesis B and

$$
0 = \text{Wh}_0(\pi_1 M) = \text{Wh}_1(\pi_1 M) = \text{Wh}_2(\pi_1 M) \otimes \mathbb{Z}[1/2] = \text{Wh}_3(\pi_1 M) \otimes \mathbb{Z}[1/2].
$$

The following theorem, which is a consequence of the parametrized surgery theory of [HS], was proved in [N3] and applies to $M$ as above:

**Theorem.** Suppose $M^m$, $m \geq 6$, is a smoothable closed aspherical manifold satisfying Hypothesis B and

$$
0 = \text{Wh}_0(\pi_1 M) = \text{Wh}_1(\pi_1 M) = \text{Wh}_2(\pi_1 M) \otimes \mathbb{Z}[1/2] = \text{Wh}_3(\pi_1 M) \otimes \mathbb{Z}[1/2] = 0.
$$

Then

(a) There is a normal abelian subgroup $H \subset \pi_0(\text{TOP}(M))$ consisting entirely of 2-torsion such that $\pi_0(\text{TOP}(M))/H \cong \text{Out}(\pi_1 M)$, where $\text{Out}(\pi_1 M)$ is the group of outer automorphisms of $\pi_1 M$.

(b) $\pi_1(\text{TOP}(M)) \otimes \mathbb{Z}[1/2] \cong \text{center}(\pi_1 M) \otimes \mathbb{Z}[1/2]$. 
Remark. Part (b) depends on the computation by K. Igusa and R. K. Dennis of the kernel of Igusa’s map \( \chi: \pi_1(P^s_{\text{diff}}(M)) \to \text{Wh}_3(\pi_1 M) \), where \( P^s_{\text{diff}}(M) \) is the space of stable smooth pseudoisotopies of \( M \) [DI].

References


[FH3] _____, Topological characterization of flat and almost flat Riemannian manifolds \( M^n (n \neq 3,4) \), Amer. J. Math. 105 (1983), 641–672.


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