PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 1, May 1984

ABELIAN p-GROUPS A AND B SUCH THAT $\operatorname{Tor}(A,G) \cong \operatorname{Tor}(B,G)$, G REDUCED

DOYLE CUTLER[1]

ABSTRACT. Let A be an abelian p-group having all of its finite Ulm invariants nonzero. Let C be a countable direct sum of cyclic p-groups such that for each nonnegative integer n, the nth Ulm invariant of C is zero if the nth Ulm invariant of A is finite. Then for all reduced abelian groups G, $\operatorname{Tor}(G,A) \cong \operatorname{Tor}(G, A \oplus C)$.

In [2], we gave an example of two nonisomorphic $p^{\omega+1}$-projective abelian p-groups A and A' with the property that for all reduced abelian groups G, $\operatorname{Tor}(G,A) \cong \operatorname{Tor}(G,A')$. In the example, $A' = A \oplus C$ where C was a countable unbounded direct sum of cyclic p-groups. In this note we will show that the same techniques provide a much more general result leading to many examples of nonisomorphic abelian p-groups A and A' such that $\operatorname{Tor}(G,A) \cong \operatorname{Tor}(G,A')$ for all reduced abelian groups G. For example, if $B = \bigoplus_{i\in\omega} B_i$ where $B_i = \bigoplus_{\aleph_0} Z(p^{i+1})$ then the torsion completion, $\overline{B}$, of B has no unbounded summand which is a direct sum of cyclic groups. Thus, if C is a countable unbounded direct sum of cyclic groups, then $\overline{B} \not\cong C \oplus \overline{B}$. We will show that $\operatorname{Tor}(G,\overline{B}) \cong \operatorname{Tor}(G, \overline{B} \oplus C)$ for all reduced abelian groups G.

In the following, all groups will be abelian groups, p is a fixed but arbitrary prime, $Z(p^n)$ is a cyclic group of order p^n, ω is the first infinite ordinal and κ is an infinite cardinal. If A is a p-group then by $r(A)$ we shall mean the rank of A. The final rank of A, denoted fin $r(A)$, is the $\inf_{n\in\omega} r(p^n A)$. The notation and terminology will be the same as that in [3].

We will need the following technical lemma in the proof of our Theorem. Recall that for a p-group A and $i \in \omega$, the ith Ulm invariant of A not zero means that A has a cyclic summand of order p^{i+1}.

LEMMA. *Let $0 \to B \to G \to \bigoplus_\kappa Z(p^\infty) \to 0$ be a pure exact sequence where κ is an infinite cardinal. Let A be p-group and $(n_i)_{i\in\omega}$ be a subsequence of ω such that the n_ith Ulm invariant of A is not zero. Then* $\operatorname{Tor}(G,A)$ *has a summand S such that $S = \bigoplus_{i\in\omega}(\bigoplus_\kappa Z(p^{n_i+1}))$.*

PROOF. By Theorem 63.2 in [3], if $0 \to B \to G \to \bigoplus_\kappa Z(p^\infty) \to 0$ is pure exact then the induced sequence $0 \to \operatorname{Tor}(B,A) \to \operatorname{Tor}(G,A) \to \bigoplus_\kappa A \to 0$ is pure exact for all p-groups A. Let λ be the induced homomorphism from $\operatorname{Tor}(G,A)$ to $\bigoplus_\kappa A$. For each $i \in \omega$, since the n_ith Ulm invariant of A is not zero, we have a decomposition $A = Z(p^{n_i+1}) \oplus A_i$. By partitioning κ into $\aleph_0$ sets of cardinality κ,

Received by the editors June 23, 1983.
1980 *Mathematics Subject Classification*. Primary 20K10, 20K40.
Key words and phrases. Abelian p-group, torsion product.
[1]This work was completed while the author was on sabbatical, visiting Tulane University.

we have $\bigoplus_\kappa A = S' \oplus A'$ where

$$S' = \bigoplus_{i\in\omega}\left(\bigoplus_{\kappa} Z(p^{n_i+1})\right) \quad\text{and}\quad A' = \bigoplus_{i\in\omega}\left(\bigoplus_{\kappa} A_i\right).$$

Since $\operatorname{Tor}(B,A)$ is pure in $\operatorname{Tor}(G,A)$, we also have $\operatorname{Tor}(B,A)$ pure in $\lambda^{-1}(S')$. By this purity and the fact that S' is a direct sum of cyclic groups, it follows from Theorem 28.2 of [**3**] that $\lambda^{-1}(S') = \operatorname{Tor}(B,A)\oplus S$ with $S \cong S'$. Since $S\cap\lambda^{-1}(A') = 0$ and $S + \lambda^{-1}(A') = \operatorname{Tor}(G,A)$, we have $\operatorname{Tor}(G,A) \cong S\oplus\lambda^{-1}(A')$.

THEOREM. *Let A be an abelian p-group such that, for all $i \in \omega$, the ith Ulm invariant of A is not zero. Let C be a countable direct sum of cyclic p-groups such that for all $i\in\omega$, the ith Ulm invariant of C is zero if the ith Ulm invariant of A is finite. Then for all reduced abelian groups G,* $\operatorname{Tor}(G,A)\cong\operatorname{Tor}(G,A\oplus C)$.

PROOF. Since A and K are p-groups, we need only consider the case in which G is a p-group. Note that if B is a basic subgroup of A, the condition on the Ulm invariants of C and the fact that C is countable implies that $B\cong B\oplus C$. It follows easily that, for all positive integers n, $A[p^n]\cong(A\oplus C)[p^n]$. Since $\operatorname{Tor}(Z(p^n),A)\cong A[p^n]$ for all groups A, we have $\operatorname{Tor}(Z(p^n),A)\cong\operatorname{Tor}(Z(p^n),A\oplus C)$. Since Tor commutes with direct sums, $\operatorname{Tor}(K,A)\cong\operatorname{Tor}(K,A\oplus C)$ for all K which are direct sums of cyclic groups.

Let G be an unbounded reduced p-group and let B be a basic subgroup of G such that $r(G/B) = \operatorname{fin} r(G)$. If $r(G/B) < r(B)$, then, by Lemma 1 of [**1**], there is a decomposition of G, say $G = H\oplus L$, such that L is a subgroup of B, $H\cap B$ is a basic subgroup of H, and $r(H/(H\cap B)) \geq r(H\cap B)$. Since Tor commutes with direct sums and $\operatorname{Tor}(L,A)\cong\operatorname{Tor}(L,A')$, we need only consider unbounded reducd p-groups G which have a basic subgroup B with $r(B)\leq r(G/B)$. With this assumption, we have the pure exact sequence $0\to B\to G\to\bigoplus_\kappa Z(p^\infty)\to 0$ where $|G| = \kappa\geq|B|$. By the above Lemma, $\operatorname{Tor}(G,A)$ has a summand $S = \bigoplus_{i\in\omega}(\bigoplus_\kappa Z(p^{i+1}))$. Note that since C is a countable direct sum of cyclic p-groups, $\operatorname{Tor}(G,Z(p^n))\cong G[p^n]$, and Tor commutes with direct sums,

$$\operatorname{Tor}(G,C) = \bigoplus_{i\in\omega}\left(\bigoplus_{\kappa_i} Z(p^{i+1})\right) \quad\text{where } \kappa_i\leq\kappa.$$

Thus $\operatorname{Tor}(G,C)\oplus S\cong S$. Hence $\operatorname{Tor}(G,A\oplus C)\cong\operatorname{Tor}(G,A)$.

COROLLARY. *Let A be an abelian p-group with the following properties:*

(1) *for $i\in\omega$, the ith Ulm invariant of A is infinite;*

(2) *A has no unbounded summand which is a direct sum of cyclic groups.*

Then for all countable unbounded direct sums of cyclic groups C, $A\not\cong A\oplus C$, but $\operatorname{Tor}(G,A)\cong\operatorname{Tor}(G,A\oplus C)$ *for all reduced abelian groups G.*

This leads one naturally to pose the following open problem.

PROBLEM. Find an example of two abelian p-groups A and A' such that for all direct sums of cyclic groups K and K', $A\oplus K\not\cong A'\oplus K'$ but $\operatorname{Tor}(G,A)\cong\operatorname{Tor}(G,A')$ for all reduced abelian groups G.

One might note that no such example exists for the class of $p^{\omega+n}$-projective p-groups. To see this, suppose that A and A' are $p^{\omega+n}$-projective p-groups such that $\operatorname{Tor}(G,A)\cong\operatorname{Tor}(G,A')$ for all reduced abelian groups G. Let $C = \bigoplus_{i\in\omega} Z(p^{i+1})$

and let H_n be a p-group such that $H_n/p^\omega H_n \cong C$ and $p^\omega H_n \cong Z(p^n)$. Let $B\,(\cong C)$ be a $p^{\omega+n-1}$-high subgroup of H_n. By Proposition 1 in [**4**], B is $p^{\omega+n}$-pure in H_n. Thus the exact sequence $0 \to B \to H_n \to Z(p^\infty) \to 0$ is $p^{\omega+n}$-pure exact. Thus by Proposition 2 in [**4**], the induced sequence $0 \to \mathrm{Tor}(A,B) \to \mathrm{Tor}(A,H_n) \to A \to 0$ is $p^{\omega+n}$-pure exact. Thus, since A is $p^{\omega+n}$-projective, $\mathrm{Tor}(A,H_n) \cong A \oplus \mathrm{Tor}(A,B)$. Since $\mathrm{Tor}(A,B)$ is a direct sum of cyclic groups and $\mathrm{Tor}(A,H_n) \cong \mathrm{Tor}(A',H_n)$, there exist direct sums of cyclic p-groups K and K' such that $A \oplus K \cong A' \oplus K'$.

References

1. K. Benabdallah and J. M. Irwin, *An application of B-high subgroups of abelian p-groups*, J. Algebra **34** (1975), 213–216.
2. D. Cutler, *Torsion products of abelian p-groups*, J. Algebra **77** (1982), 158–161.
3. L. Fuchs, *Infinite abelian groups*, Vols. 1 and 2, Academic Press, New York, 1970 and 1973.
4. R. J. Nunke, *On the structure of* Tor. II, Pacific J. Math. **22** (1967), 453–464.

Department of Mathematics, University of California, Davis, California 95616