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TOTALLY ZIPPIN p-GROUPS

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ABSTRACT. If G is a p-group of limit length λ, then it satisfies the λ-Zippin property provided that whenever $A/p^\lambda A \cong G \cong B/p^\lambda B$, every isomorphism between $p^\lambda A$ and $p^\lambda B$ extends to an isomorphism between A and B. We show that if G is almost balanced in a totally projective group, then G does satisfy the λ-Zippin property. This leads to the existence of a great variety of G's that are totally Zippin in the sense that $G/p^\alpha G$ satisfies the α-Zippin property for all limit ordinals $\alpha \leq \lambda =$ length of G. Hence totally Zippin p-groups need not be S-groups, although those of countable length turn out to be direct sums of countable groups.

Let G be an abelian p-group of limit length λ. A p-group A will be called a λ-*elongation by* G if $A/p^\lambda A \cong G$. Following Warfield [**13**], we say that G satisfies the λ-*Zippin property* if the following holds: If A and B are λ-elongations by G, then every isomorphism between $p^\lambda A$ and $p^\lambda B$ extends to an isomorphism between A and B. Zippin proved that G has this property if it is countable and Hill generalized this to the class of totally projective p-groups. In fact, Nunke [**8**] showed that the same is true for S-groups (in the sense of Warfield [**11**], and that furthermore only the Σ-cyclic p-groups enjoy the ω-Zippin property. For $\lambda > \omega$, however, examples in [**8**] suggest that nothing very definitive can be established about the structure of p-groups with the λ-Zippin property. Consequently, Warfield [**13**] proposed as a more tractable class the G's that are *totally Zippin* in the sense that $G/p^\alpha G$ has the α-Zippin property for all limit ordinals $\alpha \leq \lambda$. He moreover suggested that a reasonable conjecture would be that the totally Zippin p-groups are precisely the S-groups. This turns out not to be so and we shall, in fact, establish the existence of a considerable variety of totally Zippin p-groups of length $\Omega = \omega_1$. On a more positive note, we show that when λ is a countable limit ordinal, the only totally Zippin p-groups of length λ are the d.s.c.'s (direct sums of countable reduced p-groups); that is, Warfield's conjecture does hold for groups of countable length.

Call a subgroup H of G *almost balanced* if it is *isotype* (i.e., $p^\alpha G \cap H = p^\alpha H$ for all ordinals α) and $p^\alpha(G/H) = p^\alpha G + H/H$ for all $\alpha < \lambda =$ length of G. The isotype and λ-*dense* ($G = p^\alpha G + H$ for all $\alpha < \lambda$) subgroups H of G are just the almost balanced subgroups with G/H divisible. As the following technical result indicates, this latter class of subgroups plays a crucial role in the study of λ-elongations by G.

PROPOSITION. *Suppose A is a λ-elongation by G where λ is a limit ordinal and let D be the divisible hull of $p^\lambda A$. Then there is an isomorphism* $\theta\colon G/H \to D/p^\lambda A$,

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where H is a λ-dense, isotype subgroup of G and A can be realized as the subgroup of $G \oplus D$ consisting of all pairs (g,d), where $\theta(g+H) = d + p^\lambda A$. Conversely, any A so realized will be a λ-elongation by G.

The foregoing result evidently dates back to Kulikov [**3**] and the proof given in [**5**] for the case $\lambda = \omega$ readily generalizes.

We say that subgroups H and K of G are *congruent over* G provided $\psi(H) = K$ for some automorphism ψ of G. If furthermore for each isomorphism μ: $G/H \to G/K$ there is such an automorphism inducing μ, then we say that H and K are *strongly congruent over* G.

THEOREM 1. *Let G be a p-group of limit length λ. If each pair of λ-dense, isotype subgroups H and K with $G/H \simeq G/K$ are strongly congruent over G, then G satisfies the λ-Zippin property*

PROOF. Suppose A and B are λ-elongations by G and let ϕ: $p^\lambda A \to p^\lambda B$ be a fixed isomorphism. Without loss of generality, we may assume that $p^\lambda A$ and $p^\lambda B$ have a common divisible hull D and extend ϕ to an automorphism ϕ' of D. By our Proposition we have a pair of λ-dense, isotype subgroups H and K of G and a pair of isomorphisms θ_A: $G/H \to D/p^\lambda A$ and θ_B: $G/K \to D/p^\lambda B$ yielding representations of A and B as subgroups of $G \oplus D$. Now consider the diagram

$$\begin{array}{ccccccc} G & \xrightarrow{\pi_H} & G/H & \xrightarrow{\theta_A} & D/p^\lambda A & \xleftarrow{\pi_A} & D \\ & & \downarrow\mu & & \downarrow\sigma & & \downarrow\phi' \\ G & \xrightarrow{\pi_K} & G/K & \xrightarrow{\theta_B} & D/p^\lambda B & \xleftarrow{\pi_B} & D \end{array}$$

where σ is induced by ϕ', the π's are canonical and $\mu = \theta_B^{-1}\sigma\theta_A$. By our hypothesis on G, we have an automorphism ψ of G such that $\pi_K\psi = \mu\pi_H$. Then $\xi = \psi \oplus \phi'$ is an automorphism of $G \oplus D$ and we claim that ξ maps A onto B. Indeed if $a = (g,d) \in A$, then $\theta_A\pi_H(g) = \pi_A(d)$ and therefore

$$\theta_B\pi_K\psi(g) = \theta_B\mu\pi_H(g) = \sigma\theta_A\pi_H(g) = \theta\pi_A(d) = \pi_B\phi'(d);$$

that is, $\xi(a) = (\psi(g), \phi'(d))$ is in B. By symmetry, ξ^{-1} maps B to A and hence ξ maps A onto B.

To give some substance to Theorem 1, we quote from [**2**] the following result: If H and K are almost balanced subgroups of the totally projective p-group G of limit length, where H and K have the same Ulm invariants and $G/H \cong G/K$, then H and K are strongly congruent over G. This observation leads rather easily to

THEOREM 2. *If G has limit length λ and is an almost balanced subgroup of a totally projective p-group, then G satisfies the λ-Zippin property.*

PROOF. Let G be almost balanced in the totally projective p-group M. Replacing M by $M/p^\lambda M$ if necessary, we may assume that λ is also the length of M. Now let H and K be λ-dense, isotype subgroups of G with $G/H \cong G/K$. Since the property of being almost balanced is transitive, H and K are almost balanced in M. Take μ: $G/H \to G/K$ to be any isomorphism from G/H onto G/K. Since G/H and G/K are divisible, they are summands of M/H and M/K, respectively, with complements isomorphic to M/G. Therefore μ extends to an isomorphism

μ': $M/H \to M/K$. By the Hill-Megibben theorem quoted above, there is an automorphism ψ' of M that induces μ'. But since μ' extends μ, it follows that ψ' maps G onto itself; that is, ψ' restricts to an automorphism ψ of G that induces μ and hence H and K are strongly congruent over G.

THEOREM 3. *If G has limit length and is almost balanced in a direct sum of countable reduced p-groups, then G is a totally Zippin p-group*

PROOF. Let G be almost balanced in the d.s.c. M. By a well-known theorem of Hill's, $G/p^\alpha G$ is a d.s.c. for all $\alpha < \Omega$. In particular, if G has countable length, then it is a d.s.c. and hence totally Zippin. On the other hand, if G has length Ω, then it satisfies the Ω-Zippin property by Theorem 2 and hence is totally Zippin by our above observation on the quotients $G/p^\alpha G$.

The foregoing theorem yields a great variety of totally Zippin p-groups that fail to be S-groups. First observe by results in [**7**] that every abelian p-group is realizable as a quotient M/G where M is a d.s.c. of length Ω and G is almost balanced in M. If $p^\Omega(M/G)$ is reduced and nonzero, then G cannot be an S-group. This embarrassment of riches is, of course, due to Ω being the first limit ordinal not cofinal with ω. Matters are much simpler for ordinals cofinal with ω and, as observed earlier, we shall show there are no totally Zippin p-groups of countable length other than the d.s.c.'s. In fact if G has length λ cofinal with ω and if G is a C_λ-group (that is, $G/p^\alpha G$ is totally projective for all $\alpha < \lambda$), then G will satisfy the λ -Zippin property only if it is totally projective. Indeed using a recent theorem of Hill and the theory of C_λ-groups, we can establish this by generalizing Warfield's proof in [**12**] that only Σ-cyclic p-groups satisfy the ω-Zippin property.

THEOREM 4. *Let G be a C_λ-group of length λ, where λ is cofinal with ω. Then G satisfies the λ-Zippin property if and only if G is totally projective.*

PROOF. Assume that G is not totally projective. We shall show that there exist nonisomorphic λ-elongations A and B of G, where $p^\lambda A \cong p^\lambda B$. By Theorem 2.7 in [**10**], G contains a λ-dense, isotype subgroup H such that H is totally projective. Since G is not totally projective, the divisible group G/H is necessarily uncountable (see Theorem 1 in [**9**]). Therefore we have a direct decomposition $G/H = \bigoplus_{n=1}^{\infty} H_n/H$, where $H_n/H \cong G/H$ for all n. Then G is the union as an ascending sequence $\{G_n\}_{n<\omega}$ of λ-dense, isotype subgroups where $G_0 = H$, $G_n/H = \bigoplus_{k<n} H_k/H$ and $G/G_n \cong G/H$ for all n. By Theorem 1 in [**1**], not all of the G_n's can be totally projective. Thus G contains a λ-dense, isotype subgroup K, where $G/K \cong G/H$ and K is not totally projective. Applying our Proposition we construct a λ-elongation A by G, where $p^\lambda A$ is a direct sum of cyclic groups of order p and H is maximal in A with respect to having trivial intersection with $p^\lambda A$ (see proof of Lemma 1 in [**5**]), that is, H is a so-called p^λ-high subgroup of A. Now if L is another p^λ-high subgroup of A, then there is a height preserving isomorphism between $L[p]$ and $H[p]$ (see Lemma 6.1 in [**6**]) and the generalized Kulikov theorem in [**4**] implies that L is also totally projective. Thus all p^λ-high subgroups of A are totally projective. Another application of our Proposition yields a second λ-elongation B by G, where $p^\lambda B \cong p^\lambda A$ and K is a p^λ-high subgroup of B. Since K is not totally projective, there can be no isomorphism from A to B.

THEOREM 5. *If G is a p-group of countable limit length λ, then G is totally Zippin if and only if G is a d.s.c.*

PROOF. This follows from Theorem 4 by transfinite induction since all countable limit ordinals are cofinal with ω.

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