RESTRICTED LIE ALGEBRAS
WITH SEMILINEAR $p$-MAPPINGS

ROLF FARNSTEINER

Abstract. Restricted Lie algebras $(L, [p])$ having the property that some power of
the $p$-mapping $[p]$ is semilinear are investigated.

1. Introduction. Since restricted Lie algebras were first introduced by N. Jacobson
there have been a number of publications on the numerous correlations between
properties of the $p$-mapping and the structure of the underlying Lie algebra. It is the
main objective of this paper to determine those restricted Lie algebras $(L, [p])$ where
some power of the $p$-mapping is semilinear with respect to the corresponding power
of the Frobenius-homomorphism of the base field. It is well known that abelian
restricted Lie algebras belong to this class. Our major result illustrates that, in the
finite-dimensional case, nilpotency is a necessary condition for $[p]^n$ to be semilinear
for some $n \in \mathbb{N}$. However, the converse does not have general validity. Using our
result in combination with a commutativity theorem due to I. N. Herstein we
demonstrate that every simple associative $F$-algebra having a $p$-semilinear $p$-map is a
field.

Throughout this paper, $F$ is assumed to be a field of positive characteristic $p$. $T$
denotes an indeterminate over $F$.

Given a vector space $V$ over $F$ and a field homomorphism $\sigma: F \to F$, a mapping $f:
V \to V$ is called semilinear with respect to $\sigma$ if

$$f(\alpha x + y) = \sigma(\alpha)f(x) + f(y) \quad \forall \alpha \in F, \forall x, y \in V.$$ 

Let $L$ be a Lie algebra over $F$. We consider the Lie algebra $L \otimes_F F[T]$ and define
functions $\alpha_i^{(l)}$ by $\text{ad}_{x \otimes T^l + y \otimes 1}(x \otimes 1) = \sum_{i=0}^{l-1} \alpha_i^{(l)}(x, y) \otimes T^i$ for $l \in \mathbb{N}, x, y \in L$.

Remark 1. The following identities can be readily verified:

$$\alpha_i^{(l+1)}(x, y) = \begin{cases} \text{ad}_x(\alpha_{i-1}^{(l)}(x, y)) + \text{ad}_y(\alpha_i^{(l)}(x, y)), & i \leq l - 1, \\ \text{ad}_x(\alpha_i^{(l)}(x, y)), & i = l, \\ \text{ad}_y(\alpha_{i-1}^{(l)}(x, y)), & i = 0. \end{cases}$$

Using induction on $l$ one can check the following facts:

$$(1.1) \quad \alpha_{i-1}^{(l)}(x, y) = -\text{ad}_x(\alpha_i^{(l)}(y)).$$

Received by the editors May 31, 1983.
1980 Mathematics Subject Classification. Primary 17B50.
1Research supported by the Minna-James-Heineman Foundation.

©1984 American Mathematical Society
0002-9939/84 $1.00 + .25 per page
If $A$ is a commutative $F$-algebra we consider the Lie algebra

$$L \otimes_F A$$

and obtain $a_i^{(l)}(x \otimes a, y \otimes b) = a_i^{(l)}(x, y) \otimes a_i^{l+1}b^{l-i}$.

Let $L$ be a Lie algebra over $F$. A mapping $[p]: L \to L$ is called a $p$-mapping if

1. $\text{ad}^{[p]} = \text{ad}^p \forall x \in L$,
2. $(ax)^{[p]} = a^p x^{[p]} \forall a \in F, \forall x \in L$,
3. $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1}s_i(x, y)$ where the functions $s_i$ are given by

$$\text{ad}^{p-1}_{x \otimes T + y \otimes 1}(x \otimes 1) = \sum_{i=1}^{p-1} is_i(x, y) \otimes T^{i-1}.$$

Using methods of [3, p. 187] it can be inductively verified that given an associative algebra $A$ the identity $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1}s_i(x, y)$ holds, where $\text{ad}^{p-1}_{x \otimes T + y \otimes 1}(x \otimes 1) = \sum_{i=1}^{p-1}s_i(x, y) \otimes T^{i-1}$. Note that in contrast with the case $n = 1$ the $s_i$ are not uniquely determined in general. Let $(L, [p])$ be a restricted Lie algebra with restricted universal enveloping algebra $u(L) \supset L$. According to the above we obtain

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1}s_i(x, y) \forall x, y \in L.$$

The subsequent well-known result will be needed in the sequel.

**Lemma 1.1.** Let $(L, [p])$ be a restricted Lie algebra over $F$ and suppose that $A$ is a commutative $F$-algebra. Then there is one and only one $p$-map on $L \otimes_F A$ such that

$$(x \otimes a)^{[p]} = x^{[p]} \otimes a^p \forall x \in L \forall a \in A.$$

We conclude this section with the following

**Remark 2.** Let $A$ be an associative $F$-algebra. For $l \in \mathbb{N}$ we define functions $\lambda_i^{(l)}$ by $$(x \otimes T + y \otimes 1)^{[l]} = x^{[l]} \otimes T^l + y^{[l]} \otimes 1 + \sum_{i=1}^{l-1}\lambda_i^{(l)}(x, y) \otimes T^i, \quad l \geq 2, \quad x, y \in A.$$ A routine computation verifies the subsequent formulas:

$$\lambda_i^{(l+1)}(x, y) = \begin{cases} x\lambda_i^{(l)}(x, y) + y\lambda_i^{(l)}(x, y), & 2 \leq i \leq l - 1, \\ xy^{[l]} + yx^{[l]}(x, y), & i = 1, \\ yx^{[l]} + x\lambda_i^{(l)}(x, y), & i = l - 1. \end{cases}$$

Given a commutative $F$-algebra $B$ induction on $l$ proves

$$\lambda_i^{(l)}(x \otimes a, y \otimes b) = \lambda_i^{(l)}(x, y) \otimes a^ib^{l-i} \text{ in } A \otimes_F B.$$

**2. Conditions for semilinearity.** If $(L, [p])$ is a restricted Lie algebra over $F$ $[p]^{n}$: $L \to L$ is called semilinear if it is semilinear with respect to the homomorphism $\alpha \mapsto \alpha^{[p]}$.

**Theorem 2.1.** Let $(L, [p])$ be a restricted Lie algebra over $F$. Let $x, y$ be elements of $L$ such that $(x \otimes T + y \otimes 1)^{[p]} = x^{[p]} \otimes T^p + y^{[p]} \otimes 1$. Then $\text{ad}^{p}_{x^{[p]} - 1}(y) = 0$.

**Proof.** Consider $u(L \otimes_F F[T])$ the restricted universal enveloping algebra of $L \otimes_F F[T]$. Our assumption entails the validity of $(x \otimes T + y \otimes 1)^{[p]} = x^{[p]} \otimes T^p + y^{[p]} \otimes 1$ in $u(L \otimes_F F[T])$. From Remark 2 of the preceding section we obtain
0 = \sum_{i=1}^{p^n-1} \lambda_i(x \otimes T, y \otimes 1) = \sum_{i=1}^{p^n-1} \lambda_i(x, y) \otimes T^i. \quad \text{In the last equation we made use of the canonical identification } u(L \otimes_F F[T]) = u(L) \otimes_F F[T]. \quad \text{Hence } \lambda_i(x, y) = 0, 1 \leq i \leq p^n - 1. \text{ Note that by definition } \lambda_i(x, y) = s_i(x, y). \text{ In particular, } s_{p^n-1}(x, y) \text{ vanishes. Consequently,}

\text{ad}_{x}^{p^n-1}(y) = -\alpha_{p^n-1}(x, y) \quad \text{(cf. (1.1))}

= s_{p^n-1}(x, y) = 0.

**Corollary 2.2.** Let \((L, [p])\) be a restricted Lie algebra with center \(Z(L)\) over \(F\) such that \([p]^n\): \(L \otimes_F F[T] \to L \otimes_F F[T]\) is semilinear. Then \(L[p]^n \subset Z(L)\). If \(L\) is finite dimensional, then \(L\) is nilpotent.

**Proof.** By virtue of Theorem 2.1 we obtain \(\text{ad}_{[p]}^n = \text{ad}_{[p]} = 0\) for every \(x \in L\). \(L[p]^n\) therefore lies centrally in \(L\). In particular, \(L\) is ad-nilpotent and Engel’s Theorem yields the nilpotency of \(L\) in case of \(L\) being finite dimensional.

We are now able to give a necessary condition for the semilinearity of the mapping \([p]^n\): \(L \to L\).

**Corollary 2.3.** Let \((L, [p])\) be a restricted Lie algebra over \(F\) and let \(n\) be a positive integer such that \(\text{card}(F) > p^n\). Then the following statements hold:

1. If \(x\) and \(y\) are elements of \(L\) such that
   \[(ax + y)[p]^n = (y)[p]^n + [p]^n(y) \quad \forall a \in F,\]
   then \(\text{ad}_{x}^{p^n-1}(y) = 0.\)

2. If \([p]^n\): \(L \to L\) is semilinear, then \(L[p]^n\) is contained in \(Z(L)\) and \([p]^t\) is semilinear for \(t \geq n\).

3. If \([p]^n\) is semilinear and \(L\) is finite dimensional then \(L\) is nilpotent.

**Proof.** (1) For every \(a \in F\) there exists a homomorphism of restricted Lie algebras \(f_a\): \(L \otimes_F F[T] \to L\) such that \(f_a(x \otimes \gamma) = \gamma(a)x \quad \forall x \in L \quad \forall \gamma \in F[T].\)

According to our general definition we have \((x \otimes T + y \otimes 1)[p^n] - x[p^n] \otimes T^p - y[p^n] \otimes 1 = \sum_{i=1}^{p^n-1} s_{i}^{(n)}(x \otimes T, y \otimes 1) = \sum_{i=1}^{p^n-1} s_{i}^{(n)}(x, y) \otimes T^i.\) Applying \(f_a\) to this equation and using our present assumption yields

\begin{equation}
0 = \sum_{i=1}^{p^n-1} \alpha_{i}^{(n)}(x, y) \quad \forall \alpha \in F. \tag{*}
\end{equation}

Let \((e_j)_{j \in J}\) be a basis of \(L\) over \(F\) and write \(s_{i}^{(n)}(x, y) = \sum_{j \in J} \alpha_{i,j} e_j, 1 \leq i \leq p^n - 1.\) Put \(y_j := \sum_{i=1}^{p^n-1} \alpha_{i,j} T^i\) then \((*)\) gives

\[0 = \sum_{j \in J} y_j(a) e_j \quad \forall \alpha \in F. \quad \text{Since } \deg y_j \leq p^n - 1 \text{ we conclude } y_j = 0 \quad \forall j \in J. \quad \text{This yields } 0 = \sum_{j \in J} e_j \otimes y_j = \sum_{i=1}^{p^n-1} s_{i}^{(n)}(x, y) \otimes T^i. \quad \text{Consequently, the condition of Theorem 2.1 is satisfied by the pair } (x, y) \text{ and we obtain the desired result.}

2. The first part of our assertion is an immediate consequence of (1). Let \(t\) be an integer \(\geq n\). We write \([p]^t = [p]^{t-n} \circ [p]^n.\) By virtue of our assumption \([p]^n\) is semilinear. Since \([p]^t\) maps \(L\) into its center and \([p]^{t-n}|_{Z(L)}\) is semilinear our assertion readily follows. The last claim is a direct consequence of Engel’s Theorem.

The following example illustrates that the nilpotency of a Lie algebra does not necessarily entail the semilinearity of some power of \(p\)-mapping (cf. [4, p. 97]).
Let $V = \bigoplus_{i=1}^{p} F v_i$ be a $p$-dimensional vector space. We view $V$ as an abelian Lie algebra. Consider the endomorphism $f \in \text{End}_{F}(V)$ which is defined by means of $f(v_i) = v_{i+1}$, $1 \leq i \leq p-1$, $f(v_p) = 0$. Let $L := V \oplus Ff$ denote the semidirect product of $V$ and $Ff$, i.e. $[\alpha v + f, \beta w + g] = \alpha f(w) - \beta f(v)$ $\forall \alpha, \beta \in F$, $\forall v, w \in V$. Evidently, $L^n = f^{-1}(V)$ $\forall n \geq 2$. ($L^n$ denotes the $n$th component of the lower central series of $L$.) In particular, $L_p = F v_p$ and $L_{p+1} = 0$. The latter equation implies that $\text{ad}_f = 0$ $\forall x \in L$. Applying Theorem 11 of [3, p. 190], we conclude that there exists a $p$-mapping $[p]$: $L \to L$ such that $v_i [p] = 0$, $1 \leq i \leq p-1$, $v_j [p] = v_p$, $f [p] = 0$. Observing the identity $\text{ad}_{f^{\otimes T+v_1 \otimes f}}(f \otimes 1) = -f^n(v_1) \otimes T^{n-1}$ we find $s_i(f, v_1) = 0$, $1 \leq i \leq p-2$, and $s_{p-1}(f, v_1) = v_p$. Consequently, $(f + v_1)^{[p]} = f^{[p]} + v^{[p]}$ and $(f + v_1)^{[p]} = v_p = f^{[p]} + v^{[p]}$ $\forall n \in N$. Hence $[p]^n$: $L \to L$ is not semilinear.

An element $x \in L$ is called $p$-nilpotent if $x^{[p]} = 0$ for some $n \in N$. Note that in the preceding example the set $N(L)$ of $p$-nilpotent elements is not a subspace of $L$. Let $x$ be an element of $L$ and let $(Fx)_p$ denote the $p$-subalgebra of $L$ generated by $x$. Evidently, $(Fx)_p = \bigoplus_{i \geq 0} F x^{[p]}$. The element $x$ is called semisimple if $x \in (Fx^{[p]})_p$. If $L$ is nilpotent then every semisimple element $x \in L$ lies in the center $\mathfrak{z}(L)$.

**Theorem 2.4.** Let $(L, [p])$ be a finite-dimensional restricted Lie algebra over an infinite, perfect field. Then the following statements are equivalent:

1. There is an $n \in N$ such that $[p]^n$ is semilinear.
2. $L$ is nilpotent and $N(L)$ is a subspace of $L$.

**Proof.** (1) $\Rightarrow$ (2). According to Corollary 2.3, $L$ is nilpotent. It is clear that $N(L)$ is stable under the multiplication by scalars. Given $x, y \in N(L)$ we find $t \geq n$ such that $x^{[p]} = y^{[p]} = 0$. Owing to Corollary 2.3 $[p]^n$ is semilinear, hence $x + y \in N(L)$.

(2) $\Rightarrow$ (1). Since $F$ is perfect, every element $x$ of $L$ decomposes into a sum $x = y + z$ with $y$ semisimple and $z$ $p$-nilpotent (cf. [4, Theorem V.7.2]).

According to our previous remark, there is a subspace $V \subset \mathfrak{z}(L)$ consisting of semisimple elements such that $L = V \oplus N(L)$. (Note that the sum of two commuting semisimple elements is semisimple.) Since $N(L)$ is finite dimensional there is an $n \in N$ such that $N(L)^{[p]} = 0$. Let $x_i = y_i + z_i$, $1 \leq i \leq 2$ be elements of $L$, $y_i \in V$, $z_i \in N(L)$, $1 \leq i \leq 2$. Due to the inclusion $V \subset \mathfrak{z}(L)$ we obtain

$$
(x_1 + x_2)^{[p]} = (y_1 + y_2)^{[p]} + (z_1 + z_2)^{[p]} = y_1^{[p]} + y_2^{[p]} = x_1^{[p]} + x_2^{[p]}.
$$

Consequently, $[p]^n$ is semilinear.

B. S. Chwe has shown in [1] that if $F$ is algebraically closed then every finite-dimensional restricted Lie algebra $(L, [p])$, having the property that $0$ is the only zero of the $p$-map, is abelian. This result does not hold over arbitrary base fields. However, we can infer commutativity from semilinearity.

A restricted Lie algebra $(L, [p])$ is called $p$-algebraic if $(F x)^p$ is finite dimensional for every $x \in L$.

**Corollary 2.5.** Let $F$ be perfect such that $\text{card}(F) \geq p^n$ and suppose that $(L, [p])$ is a $p$-algebraic restricted Lie algebra such that (1) $[p]^n$: $L \to L$ is semilinear (2) $0$ is the only zero of $[p]$. Then $L$ is abelian.
RESTRICTED LIE ALGEBRAS

PROOF. Let \( x \) be an element of \( L \). Then (2) holds for the finite-dimensional restricted Lie algebra \((Fx)_p\). Since \( F \) is perfect \([p]:(Fx)_p \to (Fx)_p \) is surjective, thus \( x \in L^{[p]} \). Now Corollary 2.3 gives \( L = L^{[p]} \subseteq \beta(L) \).

3. The associative case. Every associative \( F \)-algebra \( A \) carries the structure of a restricted Lie algebra, where \( [a, b] = ab - ba \) and \( a^{[p]} = a^p \). Corollary 2.3 allows us to apply a commutativity theorem due to I. N. Herstein.

THEOREM 3.1. Let \( F \) be a field with \( \text{card}(F) \geq p^n \). Suppose that \( A \) is an associative \( F \)-algebra such that (1) \( a \mapsto a^{[p]} \) is semilinear; (2) \( A \) does not possess nontrivial nil ideals. Then \( A \) is commutative.

PROOF. Owing to Corollary 2.3, \( a^{[p]} \) lies in the center of \( A \) for every element \( a \) of \( A \). Now apply Theorem 3.2.2 of [2].

REMARK. In particular, any simple associative \( F \)-algebra with semilinear Frobenius homomorphism is a field.

ACKNOWLEDGEMENT. The author would like to take the opportunity to thank Professor G. P. Hochschild for his advice and guidance.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, University of California, Riverside, California 92521