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HYPERBOLICITY OF A COMPLEX MANIFOLD AND OTHER EQUIVALENT PROPERTIES

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ABSTRACT. Defining the notions of Schottky, Landau and Picard properties on a plane domain, the first author [3] proved that a domain in $\mathbf{C}$ having any of these properties is equivalent to the hyperbolicity of the domain.

In this paper the authors extend these notions to higher-dimensional case and obtain other various equivalent conditions for the hyperbolicity of a complex manifold.

1. Introduction. Let M be a connected complex manifold of dimension m with hermitian metric h_M. Let ρ_M denote the distance function associated with h_M. Following S. Kobayashi [**6**], we define M to be (Kobayashi) *hyperbolic* if the Kobayashi pseudometric k_M is a metric. In [**8**], H. Royden has constructed the infinitesimal form of the Kobayashi pseudometric

$$K_M(p, \xi) = \inf\{|v|: \exists f \in \mathcal{H}(\Delta, M) \ni f(0) = p, f'(0)v = \xi\} \tag{1}$$

where $(p, \xi) \in T(M)$, the tangent bundle of M, and $\mathcal{H}(\Delta, M)$ denotes the class of all holomorphic maps f of the open disc Δ into M, and he has shown that if M is hyperbolic then k_M is the integrated form of K_M. The notion of hyperbolicity of M can also be defined in terms of K_M: A complex manifold M is hyperbolic if for each $p \in M$ there exists a coordinate neighborhood U of p and a positive constant c such that for $(q,\eta) \in T(U) \simeq U \times \mathbf{C}^m$

$$K_M(q, \eta) \geqslant c\|\eta\|, \tag{2a}$$

where $\|\cdot\|$ denotes the euclidean distance. This definition of hyperbolicity coincides with that of Kobayashi (see [**8**]). In the case of a hermitian manifold, the above condition is equivalent to the condition that for each $p \in M$ there is a neighborhood U of p and a positive constant c such that

$$K_M(q, \eta) \geqslant ch_M(q,\eta) \tag{2b}$$

for all $(q, \eta) \in T(U)$.

DEFINITION 1. A complex manifold M is said to satisfy the *Schottky property* if for each $p \in M$, for each relatively compact open set W in a coordinate neighborhood of p and an $r \in (0, 1)$, there exists a positive constant $S = S(W, r)$ such that every

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holomorphic map $f \in \mathcal{H}(\Delta, M)$ with $f(0) \in W$ satisfies

$$\rho_M(p, f(z)) \leqslant S \quad \text{for } |z| \leqslant r. \tag{3}$$

M satisfies the *Landau property* if for each $p \in M$ and each relatively compact open set W in a coordinate neighborhood of p, there exists a positive constant $R = R(W)$ such that

$$\sup\{h_M(f(0), f'(0)) : f \in \mathcal{H}(\Delta, M) \text{ with } f(0) \in W\} \leqslant R. \tag{4}$$

Then we can prove the following.

THEOREM 1. *The following statements are equivalent on any hermitian manifold M.*
(a) *M is hyperbolic.*
(b) *M satisfies the Schottky property.*
(c) *M satisfies the Landau property.*

DEFINITION 2. A map $f \in \mathcal{H}(\Delta, M)$ is said to be *Bloch* (see [**4**]) if

$$\sup\{Q_f(z) : z \in \Delta\} < \infty, \tag{5a}$$

where

$$Q_f(z) = (1 - |z|^2) h_M(f(z), f'(z)). \tag{5b}$$

We remark that the nonnegative function Q_f is *invariant* under the group $\mathrm{Aut}(\Delta)$ of holomorphic automorphisms of Δ in the sense that for all $\varphi \in \mathrm{Aut}(\Delta)$

$$Q_{f \circ \varphi}(z) = Q_f(\varphi(z)), \qquad z \in \Delta. \tag{6}$$

It is an easy consequence of the definition of Kobayashi metric that if (5a) holds for all $f \in \mathcal{H}(\Delta, M)$, then M is hyperbolic. The converse is, however, not true in general (see lemma in §3). If M is compact, we obtain the following characterization of the hyperbolicity of M.

THEOREM 2. *Let M be a connected compact hermitian manifold. The following statements are equivalent.*
(a) *M is hyperbolic.*
(b) $\sup\{h_M(f(0), f'(0)) : f \in \mathcal{H}(\Delta, M)\} < \infty$.
(c) $\sup\{Q_f(z) : z \in \Delta\} < \infty$ *for all* $f \in \mathcal{H}(\Delta, M)$, *i.e., every* $f \in \mathcal{H}(\Delta, M)$ *is Bloch.*
(d) *M has the Picard property, i.e., there is no nonconstant holomorphic map f: $\mathbf{C} \to M$.*

Unlike in the one complex-dimensional case [**3**], the notion of the Picard property is a weaker notion than the Schottky property, the Landau property or the hyperbolicity for a *noncompact* manifold [**6**].

We note that the notion of the Picard property used in [**3**] and here, somewhat reluctantly, coincides with the term previously used by S. Kobayashi [**6**] and others.

Finally, we acknowledge our indebtedness to S. Krantz and also to the referee for their helpful comments on the paper. In particular, the example given in the proof of the second half of the lemma was suggested by the referee. This example is much simpler than the one originally given by the authors.

2. Proof of Theorem 1. It will be proved through the following implications: (a) $\Leftrightarrow$ (b), (a) $\Rightarrow$ (c) $\Rightarrow$ (a).

To prove (a) $\Rightarrow$ (b), let $p \in M$ and U be its local coordinate neighborhood. Let W be a relatively compact neighborhood of p in U. Since k_M is a metric, it induces the standard topology in M [**1**]. Therefore, we may define $W = \{q\colon k_M(p, q) < \rho_0\}$ for some $\rho_0 > 0$ and $U = \{q\colon k_M(p, q) < \rho_0 + \rho_1\}$, where $\rho_1 = \tan h^{-1} r$, for any given $r \in (0, 1)$. If $f \in \mathcal{H}(\Delta, M)$ satisfies $f(0) \in W$, then $k_M(p, f(0)) < \rho_0$ and $k_M(f(0), f(z)) \leqslant k_\Delta(0, z) \leqslant \rho_\Delta(0, r) = \tan h^{-1} r = \rho_1$ for $|z| \leqslant r$. Therefore, it follows from the triangle inequality that $k_M(p, f(z)) \leqslant \rho_0 + \rho_1$ whenever $|z| \leqslant r$. Since k_M is a metric, $k_M \geqslant c\rho_M$ for some $c > 0$ which proves (b) with $S = (\rho_0 + \rho_1)/c$.

For the proof of (b) $\Rightarrow$ (a), we follow the method of [**5**]: Let p and q be two distinct points of M. We may take a coordinate neighborhood U of p which does not contain q. Let $\varphi\colon U \to B = \{w \in \mathbf{C}^m\colon \|w\| < 1\}$ be a biholomorphic map with $\varphi(p) = 0$, $\varphi(U) = B$. Without loss of generality, we may assume that $\varphi(W) = \{w \in \mathbf{C}^m\colon \|w\| < \rho\} = B_\rho$ for some $\rho \in (0, 1)$ and $\varphi(V) \subset B$, $V = \{m \in M\colon \rho_M(p, M) < S\}$. Let $r \in (0, 1)$ be given. If $|z| < r/2$, then there exists a constant $c > 0$ such that $\tan h^{-1}|z| \geqslant c \tan h^{-1}|z|/r$ or, equivalently, $\rho_\Delta(0, z) \geqslant c\rho_{\Delta_r}(0, z)$. Let $\alpha = \{p = p_0, p_1, \ldots, p_l = q; a_1, \ldots, a_l; f_1, \ldots, f_l\}$ be any chain connecting p and q, used in the construction of the Kobayashi metric $k_M(p, q)$ [**6**]. Since $\varphi(q)$ lies outside B, there must be an index, say l_0, $0 \leqslant l_0 < l$, such that $\varphi(p_0), \varphi(p_1), \ldots, \varphi(p_{l_0-1}) \in B_\rho$ and $\varphi(p_{l_0}) \notin B_\rho$. By the homogeneity of Δ, we may assume that $a_1, \ldots, a_{l_0}$ lie in $\Delta_{r/2}$. Then

$$(1)\qquad |\alpha| \geqslant \sum_{i=1}^{l_0} \rho_\Delta(0, a_i) \geqslant c \sum_{i=1}^{l_0} \rho_{\Delta_r}(0, a_i) \geqslant c \sum_{i=1}^{l_0} k_B(\varphi(p_{i-1}), \varphi(p_i))$$
$$\geqslant ck_B\big(0, \varphi(p_{l_0})\big) = c \tan h^{-1}\|\varphi(p_{l_0})\| \geqslant c \tan h^{-1}\rho \equiv c'.$$

Thus, $k_M(p,q) \geqslant c' > 0$. Here we used the fact that on B the Kobayashi metric agrees with the standard Kähler metric. It is given by

$$(2)\qquad k_B(0, z) = \frac{1}{2} \log \frac{1 + \|z\|}{1 - \|z\|} = \tan h^{-1}\|z\| \quad \text{for } z \in B.$$

See [**4**].

(a) $\Rightarrow$ (c). Suppose that M fails to satisfy the Landau property. Then there exist a point p_0 in an open neighborhood W which is relatively compact in a local coordinate neighborhood U of p_0 and a sequence $f_k \in \mathcal{H}(\Delta, M)$ such that $f_k(0) \in W$ and $h_M(f_R(0), f_k'(0)) \to \infty$ as $k \to \infty$. We claim that if M is hyperbolic then there exists an $r \in (0, 1)$ such that $\{f_k\}$ contains a subsequence which converges uniformly to a holomorphic map $f\colon \Delta_r \to M$. First we observe that $\overline{W} \cap (M \setminus U) = \varnothing$ where $\overline{W}$ is compact and $M \setminus U$ is closed. Therefore, there exists a number $\rho' > 0$ such that

$$(3)\qquad Q = \{p \in M\colon k_M(\overline{W}, p) < \rho'\} \subset\subset U.$$

Since $k_M(f_k(0), f_k(z)) \leqslant \rho_\Delta(0, z) = \tan h^{-1}|z|$ for all $f_k \in \mathcal{H}(\Delta, M)$ with $f_k(0) \in W$, $f_k(z) \in Q$ whenever $|z| \leqslant r' = \tan h\rho'$. Since Q is relatively compact, it is bounded in $\mathbf{C}^m$. Therefore, by Montel's theorem there is a subsequence of $\{f_k\}$ which

converges uniformly on Δ_r to a holomorphic map $f\colon \Delta_r \to M$ for $r < r'$. Denote again this convergent subsequence by $\{f_k\}$. By Weierstrass' theorem, $f_k'(z)$ converges to $f'(z)$ uniformly on Δ_r. In particular, $f_k'(0)$ converges to $f'(0)$. Thus, $h_M(f_k(0), f_k'(0)) \to h_M(f(0), f'(0)) < \infty$ as $k \to \infty$, which is a contradiction.

(c) $\Rightarrow$ (a). Given $(q, \xi) \in T(W)$ and $v \in \mathbf{C}$, let $f \in \mathcal{H}(\Delta, M)$ satisfy $f(0) = q$ and $f'(0)v = \xi$. Then by (4) of §1

$$R|v| \geqslant h_M(f(0), f'(0))|v| = h_M(f(0), f'(0)v) = h_M(q, \xi)$$

or

$$|v| \geqslant \frac{1}{R} h_M(q, \xi), \tag{4}$$

where R is the upper bound given in (4) of §1. Thus,

$$K_M(q, \xi) \geqslant c h_M(q, \xi), \tag{5}$$

$c = 1/R$, as desired.

3. Proof of Theorem 2. By a result of R. Brody [**2**], (a) $\Leftrightarrow$ (d) holds on a connected compact manifold. Therefore, it is enough to prove the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). First we prove the following

LEMMA. *A complex manifold M is hyperbolic if* $\sup\{Q_f(z)\colon z \in \Delta\} < \infty$ *for all* $f \in \mathcal{H}(\Delta, M)$. *The converse is not true in general.*

PROOF. The above hypothesis implies that there exists a constant $Q > 0$ such that for all $f \in \mathcal{H}(\Delta, M)$

$$h_M(f(z), f'(z)v) \leqslant Q K_\Delta(z, v) \qquad (z \in \Delta, v \in \mathbf{C}). \tag{1}$$

Let $\tilde{\rho}_M$ be the distance function associated with the differential metric $\tilde{h}_M = Q^{-1} h_M$. By integrating both sides of (1) along the geodesic curve between two points z and w in Δ and using the definition of integrated distance $\tilde{\rho}_M$, we have

$$\tilde{\rho}_M(f(z), f(w)) \leqslant k_\Delta(z, w) \tag{2}$$

for all $f \in \mathcal{H}(\Delta, M)$. Since k_M is the largest among those pseudometrics which satisfy (2), $\tilde{\rho}_M \leqslant k_M$ (see [**6**]). Thus, M is hyperbolic.

To show the second half, let M be the right half-plane of $\mathbf{C}$ and h_M the euclidean metric. Then M is clearly hyperbolic. Set

$$f(z) = (1 + z)/(1 - z) \quad \text{and} \quad z_n = 1 - 1/n \in \Delta. \tag{3}$$

Then $f \in \mathcal{H}(\Delta, M)$ and

$$Q_f(z_n) = (1 - |z_n|^2)|f'(z_n)| = 2(2n - 1) \to \infty$$

as $n \to \infty$. This completes the lemma.

From this lemma, (c) $\Rightarrow$ (a) trivially follows. To prove (a) $\Rightarrow$ (b), let M be hyperbolic. By (c) of Theorem 1, for each $p \in M$ there exists a relatively compact open neighborhood W_p of p such that

$$\sup\{\|f'(0)\|\colon f \in \mathcal{H}(\Delta, M), f(0) \in W_p\} < \infty. \tag{4}$$

Since M is compact, it can be covered by a finite number of such W_p's. Thus, (b) follows.

(b) $\Rightarrow$ (c). Suppose (c) fails to hold. Then there exist sequences $\{z_n\}$ in Δ and $\{f_n\}$ in $\mathcal{H}(\Delta, M)$ such that $Q_{f_n}(z_n) > n$ for all n. Since Δ is homogeneous, for each z_n there exists $\varphi_n \in \mathrm{Aut}(\Delta)$ such that $\varphi_n(0) = z_n$. Therefore, by the invariant property of Q_f (see (6) of §1),

$$Q_{f_n}(z_n) = Q_{f_n}(\varphi_n(0)) = Q_{g_n}(0) = h_M(g_n(0), g_n'(0)), \tag{5}$$

where $g_n = f_n \circ \varphi_n$. Since h_M is hermitian, it follows from the compactness of M, condition (b) and (5) that $Q_{f_n}(z_n)$ is bounded for all n, which is a contradiction.

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