

EQUIVALENCE OF THE GREEN'S FUNCTIONS FOR DIFFUSION OPERATORS IN \mathbf{R}^n : A COUNTEREXAMPLE

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ABSTRACT. In a smooth domain in \mathbf{R}^n , the Green's functions for second-order, uniformly elliptic operators in divergence form are all proportional to the Green's function for the Laplacian [7]. In this paper we show that the above result fails for diffusion operators, that is, second-order, uniformly elliptic operators with continuous coefficients in nondivergence form. In fact, we give an example in which the Green's function is locally unbounded away from the pole.

Introduction. Our objective is to define a uniformly elliptic operator of the form

$$(1) \quad L = \sum_{i,j=1}^n a_{ij}(X) \cdot D_{X_i X_j}^2$$

with continuous coefficients in \mathbf{R}^n such that the Green's function for L in a smooth domain, D , satisfies $G(X, Y) \notin L_{\text{loc}}^{\infty}(\{(X, Y) \in D \times D: X \neq Y\})$.

Besides the nonequivalence of Green's functions for diffusion operators, the above example demonstrates a lack of smoothness in solutions of the adjoint equation, $L^*v = 0$. Here, a solution of $L^*v = 0$ in D is a function, v , in $L_{\text{loc}}^p(D)$ for some $p > 1$ such that

$$\int_D v(Y) \cdot L\phi(Y) dY = 0$$

for all $\phi \in C_0^{\infty}(D)$. We proved in [3 and 4] that nonnegative solutions of $L^*v = 0$ (in particular, the Green's function, $G(X, \cdot)$, in $D \setminus \{X\}$) are A_{∞} -weights as defined by Muckenhoupt (see [6] for several equivalent definitions). When the coefficients of L are Hölder continuous, nonnegative solutions of $L^*v = 0$ have continuous representatives [10] and satisfy a classical Harnack principle [2]. The above example shows that these results do not extend to all diffusion operators in \mathbf{R}^n and our A_{∞} -estimates of nonnegative adjoint solutions cannot be improved to L^{∞} -estimates.

1. An example in two dimensions. Let B_r denote the open ball in \mathbf{R}^2 centered at 0 with radius r . Let $D = B_6$, $\Omega = B_6 \cap \{(x, y) \in \mathbf{R}^2: y > 0\}$, and $P_0 = (0, 11/2)$. Our choice of an appropriate elliptic operator in \mathbf{R}^2 is based on the following result.

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THEOREM 1.1 (MODICA AND MORTOLA). *There exists a function, β , in $C^\infty(\Omega) \cap C(\bar{\Omega})$ such that $\frac{1}{2} \leq \beta \leq \frac{3}{2}$ in $\bar{\Omega}$, $\beta = 1$ in $\bar{\Omega} \setminus B_2$, and the operator defined by*

$$M = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} \left(\beta(x, y) \cdot \frac{\partial}{\partial y} \right)$$

has the following property: the elliptic measure at P_0 associated with M in Ω is singular with respect to surface measure on $\partial\Omega \cap B_1$.

PROOF. The elliptic measure described above, which we denote by ω^{P_0} , refers to the measure on $\partial\Omega$ characterized by the fact that if $\phi \in C(\partial\Omega)$ and u is the solution of $Mu = 0$ in Ω with $u = \phi$ on $\partial\Omega$, then $u(P_0) = \int_{\partial\Omega} \phi d\omega^{P_0}$.

L. Modica and S. Mortola [8] defined a function, α , which has all the properties listed above for β except that $\alpha \neq 1$ in $\bar{\Omega} \setminus B_2$. Choose $\beta \in C^\infty(\Omega) \cap C(\bar{\Omega})$ such that $\beta = \alpha$ in $\bar{\Omega} \cap B_1$, $\beta = 1$ in $\bar{\Omega} \setminus B_2$, and $\frac{1}{2} \leq \beta \leq \frac{3}{2}$ in $\bar{\Omega}$. Let

$$\bar{M} = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} \left(\alpha(x, y) \frac{\partial}{\partial y} \right)$$

and let $\bar{\omega}^{P_0}$ be the elliptic measure at P_0 in Ω associated with \bar{M} . Since $\beta = \alpha$ in $\bar{\Omega} \cap B_1$, it follows easily (e.g., from the comparison theorem in [5]) that $\bar{\omega}^{P_0}$ and ω^{P_0} are mutually absolutely continuous in $\partial\Omega \cap B_1$. Hence the fact that $\bar{\omega}^{P_0}$ is singular with respect to surface measure in $\partial\Omega \cap B_1$ implies the same result for ω^{P_0} .

We may extend the function, β , defined above to a continuous function in \bar{D} by setting $\beta(x, y) = \beta(x, -y)$ in \bar{D} . Let us denote this extended function by β . Set

$$L = \frac{\partial^2}{\partial x^2} + \beta(x, y) \frac{\partial^2}{\partial y^2} \quad \text{and} \quad M = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} \left(\beta(x, y) \frac{\partial}{\partial y} \right)$$

in D . We will show that the Green's function for L has the desired unboundedness in $D \times D$. We will need the following result.

Let $h(P_0; x, y) \equiv h(x, y)$ be the Green's function for M in Ω with pole at P_0 . Extend h to D by setting $h(x, -y) = -h(x, y)$ for each $Y = (x, y)$ in D .

LEMMA 1.2. *Set $v = \partial h / \partial y$ in B_5 . Then $v \in L^2(B_5)$, $L^*v = 0$ in B_5 , and the restriction of v to B_3 is not in $L^\infty(B_3)$.*

PROOF. Note that $Mh = 0$ in $\Omega \setminus P_0$, h is continuous in $\bar{\Omega} \setminus P_0$, and h is zero on $\partial\Omega \cap B_5$. It follows that $h \in H^{1,2}(B_5) \cap C(\bar{B}_5)$ and $Mh = 0$ in B_5 (see [7, p. 67]). Hence $v \in L^2(B_5)$. If $\phi \in C_0^\infty(B_5)$, we have

$$\begin{aligned} \int_{B_5} v \cdot L\phi dY &= \int_{B_5} \frac{\partial h}{\partial y} \cdot \left(\frac{\partial^2 \phi}{\partial x^2} + \beta \frac{\partial^2 \phi}{\partial y^2} \right) dx dy \\ &= \int_{B_5} \left(\frac{\partial h}{\partial x} \cdot \frac{\partial^2 \phi}{\partial x \partial y} + \beta \cdot \frac{\partial h}{\partial y} \cdot \frac{\partial^2 \phi}{\partial y^2} \right) dx dy = 0. \end{aligned}$$

Thus $L^*v = 0$ in B_5 .

Now suppose the restriction of v to B_3 is in $L^\infty(B_3)$. Choose $m > 0$ such that

$$\|v\|_{L^\infty(B_3)} = \left\| \frac{\partial h}{\partial y} \right\|_{L^\infty(B_3)} \leq m.$$

Since h is continuous in B_5 and $h = 0$ on $\partial\Omega \cap B_5$, we have

$$(2) \quad |h(x, y)| \leq m \cdot |y|$$

for all (x, y) in \bar{B}_2 . Fix $Y_0 = (x_0, 0)$ in $\partial\Omega \cap B_1$ and suppose $0 < 4r < 1$. Let $B_r(Y_0)$ denote the open ball in \mathbf{R}^2 of radius r centered at Y_0 . Then $\omega^P(B_r(Y_0) \cap \partial\Omega)$ and $h(P; x_0, r)$ are positive solutions of $Mu = 0$ in $\Omega \setminus \bar{B}_{2r}(Y_0)$ which vanish on $\partial\Omega \setminus \bar{B}_{2r}(Y_0)$. It follows from a comparison theorem (Theorem 1.4 in [5]) that

$$(3) \quad \frac{\omega^P(B_r(Y_0) \cap \partial\Omega)}{h(P; x_0, r)} \leq c_0 \cdot \frac{\omega^{(x_0, 3r)}(B_r(Y_0) \cap \partial\Omega)}{h(x_0, 3r; x_0, r)} \leq c_1$$

for all P in $\Omega \setminus B_{3r}(Y_0)$, where c_1 is independent of x_0, r and P . From (2) and (3) we deduce that

$$\omega^{P_0}(B_r(Y_0) \cap \partial\Omega) \leq c_1 \cdot mr.$$

This contradicts the fact that ω^{P_0} is singular with respect to surface measure on $\partial\Omega \cap B_1$.

For completeness, we define what we mean by the Green's function in a Lipschitz domain, D , in \mathbf{R}^n for an operator, L , as in (1). Our definition is based on the fact that for each $f \in L^p(D)$ with $p > n$, there is a unique function, u , in $H_{loc}^{2,p}(D) \cap C(\bar{D})$ such that $Lu = -f$ in D and $u = 0$ on ∂D . Pucci's estimate [9] implies that $\|u\|_{L^\infty(D)} \leq c\|f\|_{L^p(D)}$. Hence for each fixed X in D , the mapping $f \rightarrow u(X)$ is a continuous, positive linear functional on $L^p(D)$. The Riesz Representation Theorem implies the existence of a nonnegative function, $G(X, \cdot)$, in $L^{p/p-1}(D)$ such that $u(X) = \int_D G(X, Y) \cdot f(Y) dY$.

The following result is a consequence of Lemma 1.2 and some estimates on suitably "normalized" adjoint solutions proved in [3 and 4].

THEOREM 1.3. *Let $G(X, Y)$ be the Green's function for L in $D \equiv B_6$, with L as in Lemma 1.2. Then $G(X, Y) \notin L_{loc}^\infty(\{(X, Y) \in D \times D: X \neq Y\})$.*

PROOF. Let v be the solution of $L^*v = 0$ defined in Lemma 1.2. Fix X in $B_{1/4}(P_0)$ and let $\bar{v}(Y) = v(Y)/G(X, Y)$ for all Y in B_5 . Since v is not essentially bounded in B_3 , we obtain the conclusion of the theorem if we show that \bar{v} is continuous in B_3 and $\|\bar{v}\|_{L^\infty(B_3)} \leq c$ where c is independent of X .

Choose a sequence $\{\beta_j\}$, of C^∞ -functions in \mathbf{R}^2 such that $\beta_j(x, y) = \beta_j(x, -y)$ in \bar{D} , $\frac{1}{4} \leq \beta_j \leq 2$ in \bar{D} , $\beta_j = 1$ in $\mathbf{R}^2 \setminus B_2$, and $\{\beta_j\}$ converges uniformly in \bar{D} to β . Define

$$L_j = \frac{\partial^2}{\partial x^2} + \beta_j(x, y) \cdot \frac{\partial^2}{\partial y^2} \quad \text{and} \quad M_j = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} \left(\beta_j(x, y) \cdot \frac{\partial}{\partial y} \right).$$

Let $G_j(X, Y)$ be the Green's function for L_j in D . Let $h_j(P_0; Y) \equiv h_j(Y)$ be the Green's function for M_j in Ω with pole at P_0 , and extend h_j to D by setting $h_j(x, -y) = -h_j(x, y)$ for each $Y = (x, y)$ in D . Set $v_j = \partial h_j / \partial y$ in B_5 and $\bar{v}_j(Y) = v_j(Y)/G_j(X, Y)$ for all Y in B_5 . The sequence $\{h_j\}$, converges weakly in $H^{1,2}(B_5)$ to h and $\{G_j(X, \cdot)\}$ converges weakly in $L^p(B_5)$ to $G(X, \cdot)$ for $1 < p < n/(n-1)$. Since $\beta_j = \beta = 1$ in $\mathbf{R}^2 \setminus B_2$, $v_j, v, G_j(X, \cdot)$ and $G(X, \cdot)$ are harmonic functions in

$B_5 \setminus B_2$ and we may assume without loss of generality that they are continuous in $B_5 \setminus B_2$. The L^p -Schauder estimates [1] imply that $\{h_j\}$ and $\{G_j(X, \cdot)\}$ are uniformly bounded in a neighborhood of $\bar{B}_4 \setminus B_3$. Hence $\{v_j\}$ and $\{G_j(X, \cdot)\}$ converge uniformly in $\bar{B}_4 \setminus B_3$ to v and $G(X, \cdot)$, respectively. It follows that

$$\|\tilde{v}_j\|_{L^\infty(\bar{B}_4 \setminus B_3)} \leq c_0$$

where c_0 is independent of j . The constant, c_0 , is also independent of X , since the Harnack principle [11] ensures that $G_j(X_1, Y) \leq c_1 \cdot G_j(X_2, Y)$ for all X_1 and X_2 in $B_{1/4}(P_0)$ and all Y in \bar{B}_4 .

We proved in [3] (or Theorem 11.2 in [4]) that normalized adjoint solutions with respect to elliptic operators with smooth coefficients (e.g., \tilde{v}_j) satisfy the strong maximum principle; thus

$$(4) \quad \|\tilde{v}_j\|_{L^\infty(\bar{B}_4)} \leq \|\tilde{v}_j\|_{L^\infty(\partial B_4)} \leq c_0.$$

The above inequality and the Hölder estimate on normalized adjoint solutions (see [3] or Theorem 11.5 in [4]) imply that

$$\sup\{|\tilde{v}_j(Y_1) - \tilde{v}_j(Y_2)| : Y_1, Y_2 \in B_{\sigma R}(Y)\} \leq c_0 \cdot c_2 \sigma^\alpha$$

whenever $\bar{B}_{2R}(Y) \cap B_4$ and $0 < \sigma < 1$, where α and c_2 depend only on the function, β . Hence a subsequence of $\{\tilde{v}_j\}$ converges uniformly on \bar{B}_3 . The limit of the subsequence must be \tilde{v} . By (4), we have $\|\tilde{v}\|_{L^\infty(B_3)} \leq c_0$ which proves the theorem.

2. An n -dimensional example. Suppose $n > 2$ and

$$D = \{(y_1, \dots, y_n) : y_1^2 + \dots + y_{n-2}^2 < 1 \text{ and } y_{n-1}^2 + y_n^2 < 36\}.$$

Set $\gamma(y_1, \dots, y_n) = \beta(y_{n-1}, y_n)$ for all $Y = (y_1, \dots, y_n)$ in \bar{D} (where β is the function defined in §1 prior to Lemma 1.2) and set

$$L = \frac{\partial^2}{\partial y_1^2} + \dots + \frac{\partial^2}{\partial y_{n-1}^2} + \gamma(Y) \cdot \frac{\partial^2}{\partial y_n^2}$$

in \bar{D} . Then L is uniformly elliptic and the coefficients of L are continuous in \bar{D} . The following result extends Theorem 1.3 to n dimensions.

THEOREM 2.1. *The Green's function, $G(X, Y)$, for L and D as defined above satisfies $G(X, Y) \notin L_{\text{loc}}^\infty(\{(X, Y) \in D \times D : X \neq Y\})$.*

PROOF. Let $P_0 = (0, \dots, 0, 11/2)$ and

$$D_r = \{Y \in \mathbf{R}^n : y_1^2 + \dots + y_{n-2}^2 < 1 \text{ and } y_{n-1}^2 + y_n^2 < r^2\}.$$

Set $w(y_1, \dots, y_n) = v(y_{n-1}, y_n)$ for all Y in \bar{D}_5 , where v is the function defined in Lemma 1.2. Fix X in $B_{1/4}(P_0)$ and set $\tilde{w}(Y) = w(Y)/G(X, Y)$ in \bar{D}_5 . Since w depends only on y_{n-1} and y_n , Lemma 1.2 implies that the restriction of w to D_3 is not in $L^\infty(D_3)$. Hence we obtain the conclusion of the theorem if we prove that \tilde{w} is continuous in D_3 and

$$(5) \quad \|\tilde{w}\|_{L^\infty(D_3)} \leq c$$

where c is a positive constant independent of X .

Let $\{v_j\}$ and $\{\beta_j\}$ be the sequence of functions defined in the proof of Lemma 1.2. Set $w_j(y_1, \dots, y_n) = v_j(y_{n-1}, y_n)$ and $\tilde{w}_j(Y) = w_j(Y)/G_j(X, Y)$ for all Y in D_5 , where G_j is the Green's function in D for the operator

$$L_j = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_{n-1}^2} + \beta_j(y_{n-1}, y_n) \cdot \frac{\partial^2}{\partial y_n^2}.$$

Then $\{w_j\}$ converges uniformly in $\bar{D}_4 \setminus D_3$, and weakly in $L^2(D_5)$, to w . The maximum principle and Hölder estimate for normalized adjoint solutions which were used to prove Theorem 1.3 are valid in n dimensions. Thus we obtain inequality (5) by using the same line of reasoning as in Theorem 1.3.

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